

Calculus in the ring of Fermat reals

Part I: Integral calculus

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Abstract

We develop the integral calculus for quasi-standard smooth functions defined on the ring of Fermat reals. The approach is by proving the existence and uniqueness of primitives. Besides the classical integral formulas, we show the flexibility of the Cartesian closed framework of Fermat spaces to deal with infinite dimensional integral operators. The total order relation between scalars permits to prove several classical order properties of these integrals and to study multiple integrals on Peano-Jordan-like integration domains.

Keywords: Ring of Fermat reals, nilpotent infinitesimals, extension of the real field, Fermat-Reyes theorem, integration.

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1. Introduction

In spite of old controversies concerning the existence and rigor of infinitesimals, it is remarkable that using only elementary calculus, one can define and deeply study non-trivial rings containing the real field and infinitesimals. This goal has been achieved e.g. with the Levi-Civita field ([21, 22, 2, 3, 25, 26]), the Colombeau ring of generalized numbers ([4, 5, 24]) or the ring of Fermat reals ([9, 10, 12, 14, 8]).

A common erroneous opinion is that in non-Archimedean theories, we pay the price of needing a non-trivial knowledge of mathematical logic ([16]) or switching to intuitionistic logic ([18]) or using some non-trivial set theoretical methods ([6]). The non-Archimedean theories mentioned above need none of these. Another common feeling is that, to deal with infinitesimals one needs to have a great formal attitude in doing mathematics. Of course, there are half-way approaches trying to require a less formal path: e.g. at least up to a certain level, also nonstandard analysis, surely the most powerful theory of infinitesimals and infinities, can be presented without the need to face formal logic ([23]).

Aside from theories based on analysis, several algebraic methods are also available ([27, 2, 19, 1]). However, in those cases, the interested reader has to be able either to proceed formally, or to develop an intuition related to such algebraic infinitesimals. For example, in [10] it is argued that an intuitive picture of infinitesimal segments of lengths h and k such that $h^2 = k^2 = 0$ but $h \cdot k \neq 0$ is not possible. These numbers are frequently used in Synthetic Differential Geometry (SDG, [18]). At the end, it is a matter of taste about what approaches are felt as beautiful, manageable and in accordance with our philosophical approach to mathematics.

The ring $\bullet\mathbb{R}$ of Fermat reals, can be defined and studied using only elementary calculus; see ([9]). It extends the field \mathbb{R} of real numbers, and contains nilpotent infinitesimals, i.e., $h \in \bullet\mathbb{R}_{\neq 0}$ such that $h^n = 0$ for some $n \in \mathbb{N}_{>1}$. The methodological thread followed in the development of the theory of Fermat reals has been always the necessity to obtain a good dialogue between formal properties and their informal interpretations. Indeed, to cite some results of this dialogue, we can say that the ring $\bullet\mathbb{R}$ is totally ordered and geometrically representable ([10]).

Related to the ring $\bullet\mathbb{R}$, we have a good notion of smooth function called quasi-standard smooth function. These functions are arrows of the category $\bullet\mathcal{C}^\infty$ of Fermat spaces ([12, 8]), which is essentially a generalization of the

notion of diffeological space ([17]), but starting from quasi-standard smooth functions defined on arbitrary subsets $S \subseteq \bullet\mathbb{R}^s$ instead of smooth functions defined on open sets $U \subseteq \mathbb{R}^u$. The category $\bullet\mathcal{C}^\infty$ is a quasi-topos [15], and hence has several desired categorical properties, such as Cartesian closedness. Furthermore, $\bullet\mathcal{C}^\infty$ contains finite-dimensional smooth manifolds, and infinite dimensional spaces like $\mathbf{Man}(M, N)$, the space of all smooth functions between two smooth manifolds M and N , and integral and differential operators, etc. One of our goals is to develop differential geometry on these spaces using nilpotent infinitesimals. For the first steps, we need to develop differential and integral calculus, which is also preliminary to other advanced topics, such as calculus of variations, generalized functions, stochastic infinitesimals, etc.

As a start, in the present work, we develop integral calculus for quasi-standard smooth functions. Later in another paper, we will develop differential calculus. This exchange is motivated by the need of using Taylor's theorem with integral remainder in differential calculus.

In our approach, the derivatives are defined using the existence and uniqueness of the quasi-standard incremental ratio (see the Fermat-Reyes Thm. 2), and the integrals are defined using the existence and uniqueness of primitives (see Thm. 12 below). Following this approach, we have an extension of the classical results of calculus and also a quick agreement with several informal calculations used in physics (see e.g. [10]).

What are the new results following this approach, compared to classical analysis³ or NSA? Besides the possibility to use nilpotent infinitesimals to simplify several calculations, we will see that the Cartesian closed framework $\bullet\mathcal{C}^\infty$ of Fermat spaces permits to easily prove that all the classical infinite dimensional integral and differential operators are smooth maps, i.e., they are morphisms in $\bullet\mathcal{C}^\infty$. On the other hand, convenient vector spaces and related smooth functions ([7, 19]) can be embedded both in the category \mathbf{Dlg} of diffeological spaces and in the category $\bullet\mathcal{C}^\infty$ of Fermat spaces (see [12, 17]). Therefore, one of the best calculus in locally convex topological vector spaces is still available in $\bullet\mathcal{C}^\infty$. Finally, the Fermat functor $\bullet(-) : \mathbf{Dlg} \longrightarrow \bullet\mathcal{C}^\infty$ ([8, 15]) connects these categories and has very good preservation properties.

What are the new results compared to smooth infinitesimal analysis (SIA; see e.g. [20])? In the ring of Fermat reals, we have a total order relation rather than only a pre-order, and indeed we are able to prove classical inequalities and order properties of integrals. Because of the simplicity of the whole model (if compared to those of SIA), in our framework we can not only formally repeat several proofs of SIA, but also suitably generalize

³Here by classical analysis, we mean the most developed theory of infinite dimensional spaces, i.e., locally convex topological vector space theory. We recall that the category of these spaces is not Cartesian closed; see [11].

some classical existence theorems like the inverse function theorem and the intermediate value theorem.

The structure of the paper is as follows.

In Section 2, we summarize the basics of Fermat reals: unique decomposition of elements, Taylor’s formula with nilpotent increments, Fermat topology, extension of ordinary smooth functions, well-ordering, quasi-standard smooth functions, etc.

In Section 3, we review another topology on Fermat reals called the ω -topology ([14]), which is only used in this section. Unlike the Fermat topology, we show that in general, quasi-standard smooth functions are not continuous with respect to this topology. On the other hand, we prove a uniqueness theorem (Thm. 5), which says that every quasi-standard smooth function is determined by its values on any dense subset of the domain in the ω -topology. This uniqueness theorem is a generalization of previous results like the cancellation law of non-infinitesimal functions ([12, Lem. 25]), and it will be useful for subsequent sections.

In Section 4, we show that every interval and its interior in ${}^\bullet\mathbb{R}$ is connected (Lem. 6), and we use this to prove the constant function theorem (Thm. 7), i.e., that only constant quasi-standard smooth functions have zero derivatives on a non-infinitesimal interval (see Sec. 2). From now on, we require the ambient quasi-standard smooth function to be defined on a non-infinitesimal interval, due to the uniqueness of the incremental ratio in the Fermat-Reyes theorem (Thm. 2). Since the domain of every quasi-standard smooth function defined on an interval of ${}^\bullet\mathbb{R}$ can always be extended to a Fermat open set (although the extension may not be unique; see Lem. 8), we generalize the Fermat-Reyes theorem for quasi-standard smooth functions defined on a non-infinitesimal interval instead of a Fermat open set, and hence introduce the notion of left and right derivatives (Lem. 10).

In Section 5, we prove the existence and uniqueness of primitives for quasi-standard smooth functions (Thm. 12 and Thm. 13) by the sheaf property of such functions and the constant function theorem (Thm. 7). From this, we define integrals of quasi-standard smooth functions (Def. 14), which is a natural generalization of integrals of ordinary smooth functions (Rem. 15). Note that, although we require the ambient quasi-standard smooth function to be defined on a non-infinitesimal interval, the integrals can be defined on an infinitesimal interval. From the proof of Thm. 12, we get a useful expression of integrals (Lem. 16) as a sum of integrals of the local expression of quasi-standard smooth functions as extension of ordinary smooth functions with parameters.

Although most results about integrals of quasi-standard smooth functions can be proved using Lem. 16, the local expression, (see Rem. 18), we prefer “global” proofs. In Section 6 and the first part of Section 7, we prove the classical formulas of integrals calculus (Thm. 17, Thm. 19, Lem. 23 and Cor. 24). The “global” proof of the additivity of integrals with respect to

the integration intervals (Cor. 24) relies on the commutativity of differentiation and integration (Lem. 22), and the latter is a consequence of the $\bullet\mathcal{C}^\infty$ smoothness of some infinite dimensional integral operators (Lem. 20 and Cor. 21). Finally, we get a form of mean value theorem called Hadamard's lemma (Cor. 25).

In the rest of Section 7, we study the standard and infinitesimal parts of an integral (Def. 26 and Lem. 27), and we review divergence and curl using suitable (nilpotent) infinitesimals in Fermat reals. As a result, any integral over an infinitesimal integration interval is infinitesimal, and any integral of an infinitesimal-valued quasi-standard smooth function is infinitesimal.

Thanks to the total order on $\bullet\mathbb{R}$, in Section 8, we get some similar inequalities for our integrals as the classical ones, with respect to arbitrary integration interval (monotonicity of integral (Lem. 28 and Thm. 29), a replacement of the inequality for absolute value of quasi-standard smooth functions (Thm. 30), and the Cauchy-Schwarz inequality (Thm. 31)).

In Section 9, we study multiple integrals of quasi-standard smooth functions, with integration domain a finite family of pairwise disjoint boxes (Thm. 34 and Thm. 37), and we prove Fubini's theorem (Thm. 36).

2. Background for Fermat reals

In this section, we summarize the basics of Fermat reals by listing some properties that permit to characterize the ring $\bullet\mathbb{R}$ (see [13] for a formal list of axioms). For a more detailed presentation and for a description of the very simple model, see [9, 8]. See also [14] for the study of $\bullet\mathbb{R}$ using metrics, the characterization of its ideals, roots of nilpotent infinitesimals, some applications to fractional derivatives, and a computer implementation.

At the end of this section, we fill in a gap in the proof of the Fermat-Reyes theorem in [12], so that the statement of the theorem (see Thm. 2) is slightly different, but the differential calculus developed in [12] still holds.

Fermat reals can be defined as a quotient ring of the ring of little-oh polynomials. Equivalently, every Fermat real $x \in \bullet\mathbb{R}$ can be written, in a unique way, as

$$x = {}^\circ x + \sum_{i=1}^N \alpha_i \cdot dt_{a_i}, \quad (1)$$

where ${}^\circ x, \alpha_i, a_i \in \mathbb{R}$ are standard reals, $a_1 > a_2 > \dots > a_N \geq 1$, $\alpha_i \neq 0$, and dt_a verifies the following properties:

$$\begin{aligned} dt_a \cdot dt_b &= dt_{\frac{ab}{a+b}} \\ (dt_a)^p &= dt_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\geq 1} \\ dt_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned}$$

The expression (1) is called the *decomposition* of x , and the real number ${}^\circ x$ its *standard part*. The greatest number $a_1 =: \omega(x)$ is called the *order* of x and represents the greatest infinitesimal appearing in its decomposition. When $x \in \mathbb{R}$, i.e., $x = {}^\circ x$, we set $\omega(x) := 0$. We will also use the notations $\omega_i(x) := a_i$ and ${}^\circ x_i := \alpha_i$ for the i -th order and the i -th standard part of x . The order $\omega(-)$ has the following expected properties:

$$\begin{aligned}\omega(x + y) &\leq \max\{\omega(x), \omega(y)\} \\ \frac{1}{\omega(x \cdot y)} &= \frac{1}{\omega(x)} + \frac{1}{\omega(y)} \quad \text{if } {}^\circ x = {}^\circ y = 0 \text{ and } x \cdot y \neq 0.\end{aligned}$$

Using the decomposition of x , it is not hard to prove that for $k \in \mathbb{N}_{>1}$, $x^k = 0$ iff $\omega(x) < k$. For $k \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the ideal

$$D_k := \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0, \omega(x) < k + 1\}$$

plays a fundamental role in k -th order Taylor's formula with nilpotent increments (so that the remainder is zero). Indeed, for $k \in \mathbb{N}_{\geq 1}$ we have that $D_k = \{x \in {}^\bullet\mathbb{R} \mid x^{k+1} = 0\}$. We simply write D for D_1 . Every ordinary smooth function $f : A \rightarrow \mathbb{R}$, defined on an open set A of \mathbb{R}^d , can be extended to a function ${}^\bullet f : {}^\bullet A \rightarrow {}^\bullet\mathbb{R}$ defined on

$${}^\bullet A := \{x \in {}^\bullet\mathbb{R}^d \mid {}^\circ x \in A\},$$

preserving old values $f(x) \in \mathbb{R}$ for $x \in A$. Note that, for A, B open sets of Euclidean spaces, we can naturally identify ${}^\bullet(A \times B)$ with ${}^\bullet A \times {}^\bullet B$, and hence there is no ambiguity to write ${}^\bullet\mathbb{R}^d$ for $({}^\bullet\mathbb{R})^d$ or ${}^\bullet(\mathbb{R}^d)$; see [12, Thm. 19]. The mentioned Taylor's formula is therefore

$$\forall h \in D_k^d : {}^\bullet f(x + h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \leq k}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x), \quad (2)$$

where $x \in A$, and $D_k^d = D_k \times \dots \times D_k$.

There is a natural topology on ${}^\bullet\mathbb{R}^d$ consisting of sets of the form ${}^\bullet A$ for open sets $A \subseteq \mathbb{R}^d$. We call this topology the *Fermat topology*, and open sets in the Fermat topology the *Fermat open sets*. Unless we specify otherwise, in the present work we always equip every subset of ${}^\bullet\mathbb{R}^d$ with the Fermat topology when viewed as a topological space.

It may seem difficult to work in a ring with zero divisors. But the following properties permit to deal effectively with products of nilpotent infinitesimals (typically appearing in Taylor's formula of several variables) and with cancellation law:

$$h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0 \quad \Longleftrightarrow \quad \sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$$

$$x \text{ is invertible} \iff {}^\circ x \neq 0$$

$$(\text{If } x \cdot r = x \cdot s \text{ in } {}^\bullet\mathbb{R}, \text{ where } r, s \in \mathbb{R} \text{ and } x \neq 0) \implies r = s.$$

The (commutative unital) ring ${}^\bullet\mathbb{R}$ is totally ordered, and the order relation can be effectively decided starting from the decompositions of elements as follows. For $x \in {}^\bullet\mathbb{R}$, if ${}^\circ x \neq 0$, then $x > 0 \iff {}^\circ x > 0$; otherwise, if ${}^\circ x = 0$, then $x > 0 \iff {}^\circ x_1 > 0$.

For example, $dt_3 - 3dt > dt > 0$, and $dt_a < dt_b$ if $a < b$. In the following, we will use symbols like $[a, b] := \{x \in {}^\bullet\mathbb{R} \mid a \leq x \leq b\}$ for intervals in ${}^\bullet\mathbb{R}$, whereas intervals in \mathbb{R} will be denoted by symbols like $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$. By an *interval* in ${}^\bullet\mathbb{R}$ we mean a set of the form $[a, b]$, (a, b) , $[a, b)$ or $(a, b]$, where $a \leq b$. An interval is called *non-infinitesimal* if ${}^\circ a < {}^\circ b$. Since \leq is a total order on ${}^\bullet\mathbb{R}$, it is easy to prove e.g. that $a = \inf(a, b)$, $b = \sup(a, b)$. Therefore, each interval of ${}^\bullet\mathbb{R}$ uniquely determines its endpoints.

The calculus we begin to develop in the present work concerns quasi-standard smooth functions, which were introduced in [12]. Recall that a function $f : S \rightarrow T$ with $S \subseteq {}^\bullet\mathbb{R}^s$ and $T \subseteq {}^\bullet\mathbb{R}^t$, is called *quasi-standard smooth* if locally (in the Fermat topology) it can be written as

$$f(x) = {}^\bullet\alpha(p, x) \quad \forall x \in {}^\bullet V \cap S, \quad (3)$$

where $\alpha \in \mathcal{C}^\infty(P \times V, \mathbb{R}^t)$ is an ordinary smooth function defined on an open set $P \times V \subseteq \mathbb{R}^p \times \mathbb{R}^s$, and $p \in {}^\bullet P$ is some fixed p -dimensional parameter⁴. Every quasi-standard smooth function is continuous when both domain and codomain are equipped with the Fermat topology (see [12, Thm. 13]). We write $\mathbf{S}^\bullet\mathbb{R}^\infty(A, B)$ for the set of all quasi-standard smooth functions from $A \subseteq {}^\bullet\mathbb{R}^a$ to $B \subseteq {}^\bullet\mathbb{R}^b$. Moreover, when A and B are open sets of Euclidean spaces, $\mathbf{S}^\bullet\mathbb{R}^\infty(A, B) = {}^\bullet\mathcal{C}^\infty(A, B) = \mathcal{C}^\infty(A, B)$ (see [12, Thm. 18(4)]). Therefore, the notions of “ordinary smooth function” (i.e., functions such that all iterated partial derivatives exist and are continuous), of “quasi-standard smooth function” and of “arrow of the category ${}^\bullet\mathcal{C}^\infty$ ” (see [12, Def. 16]) are consistent. From now on, we call every such function a *smooth function*, and one can tell which kind it is from its domain and codomain.

In [12], the Fermat-Reyes theorem, which is essential for the development of the differential calculus on ${}^\bullet\mathbb{R}$, was presented. Unfortunately, there is a gap in formula (22) of [12], since nothing guarantees that the neighborhood ${}^\bullet B$ of h is sufficiently small for that formula to hold. Nevertheless, it is clear that we need to guarantee the validity of the Fermat-Reyes formula (see (6) below) only for h sufficiently small. In the following definition, we precise the meaning of the sentence “for h sufficiently small, the property $\mathcal{P}(h)$ holds”.

⁴Note the different fonts used e.g. for the point $p \in P$ and the dimension of $\mathbb{R}^p \supseteq P$.

Definition 1. Let $\mathcal{P}(h)$ be a property of $h \in S \subseteq \bullet\mathbb{R}$. We write

$$\forall^0 h \in S : \mathcal{P}(h),$$

and we read it as *for $h \in S$ sufficiently small $\mathcal{P}(h)$ holds*, if

$$\exists \rho \in \mathbb{R}_{>0} \forall h \in S : |h| < \rho \Rightarrow \mathcal{P}(h).$$

Note explicitly that this notation also includes the special case $S \subseteq \mathbb{R}$. Note also that, in a formula like $\forall x \forall^0 h : \mathcal{P}(x, h)$, if we say that $\mathcal{P}(x, h)$ holds for $|h| < \rho$, then ρ depends on x .

For example, if $U \subseteq \bullet\mathbb{R}^d$ is a Fermat open set and $v \in \bullet\mathbb{R}^d$, then

$$\forall x \in U \forall^0 h \in \bullet\mathbb{R} : x + hv \in U. \quad (4)$$

Using this language, the new statement of the Fermat-Reyes theorem is the following. The above notation $\forall^0 h$ is very convenient whenever thickenings (see [12, Lem. 23].) are implicit, as in the following:

Theorem 2. *Let U be a Fermat open set of $\bullet\mathbb{R}$, and let $f : U \rightarrow \bullet\mathbb{R}$ be a smooth function. Then there exists one and only one smooth map $r \in \bullet\mathcal{C}^\infty(\tilde{U}, \bullet\mathbb{R})$ such that*

$$\forall (x, h) \in \tilde{U} : f(x + h) = f(x) + h \cdot r(x, h) \quad \text{in } \bullet\mathbb{R}. \quad (5)$$

Without citing the thickening \tilde{U} , we can state (5) as

$$\forall x \in U \forall^0 h \in \bullet\mathbb{R} : f(x + h) = f(x) + h \cdot r(x, h) \quad \text{in } \bullet\mathbb{R}. \quad (6)$$

We can thus define $f'(x) := r(x, 0) \in \bullet\mathbb{R}$ for every $x \in U$. Moreover, if $f(x) = \bullet\alpha(p, x)$, $\forall x \in \bullet V \subseteq U$ with $\alpha \in \mathcal{C}^\infty(P \times V, \mathbb{R})$, then

$$f'(x) = \bullet \left(\frac{\partial \alpha}{\partial x} \right) (p, x) \quad \text{in } \bullet\mathbb{R}.$$

Proof. Take $(x, h) \in \tilde{U}$. We first consider the case ${}^\circ h = 0$. Since $x \in U$, we can write $f|_{\mathcal{V}_x} = \bullet\alpha_x(p_x, -)|_{\mathcal{V}_x}$, where $\alpha_x \in \mathcal{C}^\infty(P_x \times \bar{V}_x, \mathbb{R})$, $\mathcal{V}_x := \bullet\bar{V}_x \subseteq U$ is a Fermat open neighborhood of x and $\bullet P_x$ is a Fermat open neighborhood of some fixed parameter $p_x \in \bullet\mathbb{R}^{p_x}$. Set $T_x := \bullet\bar{V}_x$ for the thickening of $\bullet\bar{V}_x$. Since T_x is a Fermat open subset of $\bullet\mathbb{R}^2$ (see [12, Lem. 23]), we can find two open subsets A, B of \mathbb{R} such that $\bullet(A \times B) \subseteq T_x$, ${}^\circ x \in A$ and ${}^\circ h = 0 \in B$. Since \tilde{U} is also a Fermat open set, without loss of generality, we can assume that $\bullet(A \times B) \subseteq \tilde{U}$. By definition of thickening, we have $a + s \cdot b \in \bar{V}_x$ for all $a \in A$, $b \in B$ and $s \in [0, 1]_{\mathbb{R}}$. This proves correctly property (22) of [12]. From this point on, we can continue like in [12], proving that we can define a smooth function $r : \bullet(A \times B) \rightarrow \bullet\mathbb{R}$ that satisfies (5) in its

domain. If ${}^\circ h \neq 0$, then h is invertible, and hence we can simply define $r(x, h) := \frac{f(x+h)-f(x)}{h}$ in a neighborhood ${}^\bullet(A \times B) \subseteq \tilde{U}$ of (x, h) such that $0 \notin B$. Using the sheaf property in ${}^\bullet\mathcal{C}^\infty$, exactly like in [12], we have the conclusion. \square

Here is a simple application. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function, then $({}^\bullet g)'(x) = {}^\bullet(g')(x)$. The Fermat-Reyes theorem hence implies that the function $\mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$(s, t) \mapsto \begin{cases} \frac{g(s+t)-g(s)}{t}, & \text{if } t \neq 0 \\ g'(s), & \text{if } t = 0 \end{cases}$$

is smooth.

Note that the Fermat-Reyes theorem permits to differentiate smooth functions only at interior points (in the Fermat topology).

3. A uniqueness theorem

Before discussing integration of quasi-standard smooth functions, we prove in this section a very useful uniqueness theorem.

In order to state the theorem, we need another topology on Fermat reals, called the ω -topology. We summarize the basics of the ω -topology; see [14] for more details.

Let $d_\omega : {}^\bullet\mathbb{R}^n \times {}^\bullet\mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be defined by $d_\omega(x, y) = \|{}^\circ x - {}^\circ y\| + \sum_{i=1}^n \omega(x_i - y_i)$. Then d_ω is a complete metric on ${}^\bullet\mathbb{R}^n$, and the topology induced by this metric is called the ω -topology, which is strictly finer than the Fermat topology. For any $x \in {}^\bullet\mathbb{R}^n$ and any $s \in (0, 1]_{\mathbb{R}}$, every element in the open ball $B_s(x; d_\omega)$ is of the form $x + r$ for $r \in \mathbb{R}^n$ with $\|r\| < s$ (see [14, Thm. 7, 8, 11]). We will denote by A_ω the topological space given by a subset A of ${}^\bullet\mathbb{R}^n$ with the ω -topology. Note that for U, V open sets of Euclidean spaces, not every ordinary smooth function $f : U \rightarrow V$ induces a continuous function ${}^\bullet f : {}^\bullet U_\omega \rightarrow {}^\bullet V_\omega$. For example, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is the square function $f(x) = x^2$, then $({}^\bullet f)^{-1}(B_1(dt; d_\omega)) =: X$ is not open in ${}^\bullet\mathbb{R}_\omega$. In fact, one can check that $dt_2 \in X$, but no ω -open neighborhood of it is contained in X .

Remark 3.

- (i) The set of all invertible elements in ${}^\bullet\mathbb{R}$ is open and dense in both ${}^\bullet\mathbb{R}_\omega$ and ${}^\bullet\mathbb{R}$, where the latter has the Fermat topology.
- (ii) The ideal D_∞ is closed in both ${}^\bullet\mathbb{R}_\omega$ and ${}^\bullet\mathbb{R}$. More generally, every ideal in ${}^\bullet\mathbb{R}$ is closed in ${}^\bullet\mathbb{R}_\omega$, but only the (unique) maximal (prime) ideal D_∞ is closed in ${}^\bullet\mathbb{R}$.
- (iii) The sub-topologies on \mathbb{R}^n via the embeddings $\mathbb{R}^n \rightarrow {}^\bullet\mathbb{R}^n$ and $\mathbb{R}^n \rightarrow {}^\bullet\mathbb{R}_\omega^n$ both coincide with the Euclidean topology.

The following lemma is a trivial consequence of the uniqueness of decomposition of Fermat reals:

Lemma 4. *Assume that $x \in \bullet\mathbb{R}$ can be written as $x = r + \sum_{i=1}^N \alpha_i \cdot dt_{a_i}$, where $N \in \mathbb{N}$, $r, \alpha_1, a_1, \dots, \alpha_N, a_N \in \mathbb{R}$ and $a_1 > \dots > a_N \geq 1$. Then $x = 0$ implies $r = 0$ and $\alpha_i = 0$ for all $i = 1, \dots, N$.*

Proof. Since $r = {}^\circ x = 0$, we have the first part of the conclusion. If $N = 0$, the second part is trivial. Otherwise, $x = \sum_{\substack{i=1, \dots, N \\ \alpha_i \neq 0}} \alpha_i \cdot dt_{a_i}$ is the decomposition of x , and the conclusion then follows from the uniqueness of the decomposition. \square

A writing of the form $x = r + \sum_{i=1}^N \alpha_i \cdot dt_{a_i}$ satisfying the conditions of this lemma (compared to the decomposition of x , we allow α_i 's to be 0) will be called a *quasi-decomposition* of x .

Here is the uniqueness theorem:

Theorem 5. *Let A be an arbitrary subset of $\bullet\mathbb{R}^n$, and let $f, g : A \rightarrow \bullet\mathbb{R}$ be smooth functions. If $f(x) = g(x)$ for all x in a dense subset of A_ω , then $f(x) = g(x)$ for all $x \in A$.*

Proof. We may assume that g is the zero function. For any $a \in A$ there exist an open neighborhood V of ${}^\circ a$ in \mathbb{R}^n , an open set $P \subseteq \mathbb{R}^p$, a fixed parameter $p \in P$, and an ordinary smooth function $\alpha : P \times V \rightarrow \mathbb{R}$ such that $f(x) = \bullet\alpha(p, x)$ for all $x \in \bullet V \cap A$.

Let $p = (p_1, \dots, p_p) \in \bullet\mathbb{R}^p$ and $a = (a_1, \dots, a_n) \in \bullet\mathbb{R}^n$ be the components of p and a . Let $p_l = {}^\circ p_l + \sum_{i=1}^{N_l} p_{il} dt_{q_{il}} =: {}^\circ p_l + k_l$, $l = 1, \dots, p$, and $a_m = {}^\circ a_m + \sum_{i=1}^{M_m} a_{im} dt_{b_{im}} =: {}^\circ a_m + h_m$, $m = 1, \dots, n$, be the decompositions of these components with k_l, h_m their infinitesimal parts. For $r \in \mathbb{R}^n$ such that $\|r\| < 1$ and $a + r \in \bullet V \cap A$, we can write

$$f(a + r) = \bullet\alpha(p, a + r) = \bullet\alpha({}^\circ p_1 + k_1, \dots, {}^\circ p_p + k_p, {}^\circ x_1 + h_1, \dots, {}^\circ x_n + h_n),$$

where $x_m = {}^\circ a_m + r_m$ for $m = 1, \dots, n$. We show below that the coefficients in a quasi-decomposition of $f(a + r)$ are ordinary smooth functions of r . From the above decomposition, we have $k_l \in D_{q_{il}}$ and $h_m \in D_{b_{im}}$. Thereby, denoting by $\lceil u \rceil \in \mathbb{N}$ the ceiling of $u \in \mathbb{R}$ and setting $k := (k_1, \dots, k_p)$, $h := (h_1, \dots, h_n)$ and $c := \max_{l=1, \dots, p} (\lceil q_{il} \rceil, \lceil b_{1m} \rceil)$, we have that $(k, h) \in D_c^{p+n}$.

Using Taylor's formula (2) for the smooth function α at the point $({}^\circ p, {}^\circ a + r)$ and with increments (k, h) , we obtain

$$f(a + r) = \bullet\alpha(p, a + r) = \sum_{j \in J} \partial_j \alpha({}^\circ p, {}^\circ a + r) \cdot \frac{(k, h)^j}{j!}, \quad (7)$$

where J is the set of all the multi-indices $j \in \mathbb{N}^{\mathbf{p}+n}$ such that $|j| \leq c$. Since we have that

$$(k, h)^j = \prod_{l=1}^{\mathbf{p}} k_l^{j_l} \cdot \prod_{m=1}^n h_m^{j_{\mathbf{p}+m}} = \prod_{l=1}^{\mathbf{p}} \left(\sum_{i=1}^{N_l} p_{il} dt_{q_{il}} \right)^{j_l} \cdot \prod_{m=1}^n \left(\sum_{\iota=1}^{M_m} a_{\iota m} dt_{b_{\iota m}} \right)^{j_{\mathbf{p}+m}},$$

i.e., a finite product of elements in the ideal D_c , the decomposition of $(k, h)^j$ can be written as

$$(k, h)^j = \sum_{e=1}^{K_j} \gamma_{je} \cdot dt_{s_{je}} \quad (8)$$

for some constants $K_j \in \mathbb{N}$, $\gamma_{je} \in \mathbb{R}$ and $s_{je} \in \mathbb{R}_{\geq 1}$ (depending on the fixed datum a and f , of course). For simplicity, we can also use (8) for $j = 0 \in \mathbb{N}^{\mathbf{p}+n}$ if we set $K_j = \gamma_{je} = 1$, $s_{je} = 0$ and $dt_0 = 1$. Substituting (8) into (7), we get

$$f(a+r) = \bullet \alpha(p, a+r) = \sum_{j \in J} \sum_{e=1}^{K_j} \partial_j \alpha(\circ p, \circ a+r) \cdot \frac{1}{j!} \cdot \gamma_{je} \cdot dt_{s_{je}}.$$

To arrive at a quasi-decomposition of $f(a+r)$, it remains to gather all the summands having the same term $dt_{s_{je}}$. Set

$$L := \{s_{je} \mid j \in J, e = 1, \dots, K_j\},$$

let $\{w_1, \dots, w_H\} := L$ be an enumeration of L such that $w_1 > \dots > w_H$, and finally, for each $\ell = 1, \dots, H$, set

$$C_\ell(r) := \sum \left\{ \partial_j \alpha(\circ p, \circ a+r) \cdot \frac{1}{j!} \cdot \gamma_{je} \mid s_{je} = w_\ell, j \in J, e = 1, \dots, K_j \right\}.$$

Then

$$f(a+r) = \sum_{\ell=1}^H C_\ell(r) \cdot dt_{w_\ell} \quad (9)$$

is a quasi-decomposition of $f(a+r)$, and each $C_\ell(-)$ is an ordinary smooth function of r in a neighborhood of $0 \in \mathbb{R}^n$. The assumption implies that we can find a real sequence $(r(v))_{v \in \mathbb{N}} \downarrow 0$ such that $a+r(v) \in \bullet V \cap A$ and $f(a+r(v)) = 0$ for all $v \in \mathbb{N}$. Therefore, $\sum_{\ell=1}^H C_\ell(r(v)) \cdot dt_{w_\ell} = 0$ for all v . Lem. 4 yields that $C_\ell(r(v)) = 0$ for all ℓ , and hence $C_\ell(0) = 0$ by the continuity of $C_\ell(-)$. Setting $r = 0$ in (9), we get the conclusion that $f(a) = 0$. \square

This theorem will be used frequently in the following sections.

4. The constant function theorem

Our first problem concerns the existence and uniqueness of primitives of smooth functions $f : [a, b] \rightarrow \bullet\mathbb{R}$, and hence the starting of an integration theory. We will solve this problem in next section, and in this section, we first prove a useful instrument for solving this problem, namely the constant function theorem.

In the following, we use simplified symbols like $[a, b]$ also to denote the corresponding subspace of $\bullet\mathbb{R}$ in the category $\bullet\mathcal{C}^\infty$, i.e., for $([a, b] \prec \bullet\mathbb{R})$ (see [12] for the notation $(X \prec Y)$). We set $\bullet\mathbb{R} := \bullet\mathbb{R} \cup \{+\infty, -\infty\}$ with the usual operations, and $^\circ a = a$ if $a \in \{\pm\infty\}$, even if we will never consider intervals closed at $-\infty$ or $+\infty$.

It is not hard to prove that if $a, b \in \mathbb{R}$, then

$$\begin{aligned} \bullet\{(a, b)_{\mathbb{R}}\} &\subsetneq (a, b) \\ \bullet\{[a, b]_{\mathbb{R}}\} &\subsetneq [a, b]. \end{aligned}$$

For example, $x_t := a - t^2$ is equal to a in $\bullet\mathbb{R}$, and hence it belongs to the interval $[a, b]$, but $x \notin \bullet\{[a, b]_{\mathbb{R}}\}$ because x does not map $\mathbb{R}_{\geq 0}$ into $[a, b]_{\mathbb{R}}$. Analogously, $a + dt \in (a, b)$ but $^\circ(a + dt) = a \notin (a, b)_{\mathbb{R}}$, so $a + dt \notin \bullet\{(a, b)_{\mathbb{R}}\}$. This implies that the interval (a, b) , with $a, b \in \bullet\mathbb{R}$, is not open in the Fermat topology if at least one of the endpoints is finite, and thus we are initially forced to consider derivatives only at its interior points, where

$$\begin{aligned} \text{int}\{\bullet\{(a, b)_{\mathbb{R}}\}\} &= \text{int}\{\bullet\{[a, b]_{\mathbb{R}}\}\} = \text{int}[a, b] = \text{int}(a, b) \\ &= \{x \in \bullet\mathbb{R} \mid ^\circ a < ^\circ x < ^\circ b\}. \end{aligned}$$

Here, of course, $\text{int}(A)$ denotes the interior of A in the Fermat topology, and we have used simplified notations like $\text{int}[a, b] = \text{int}\{[a, b]\}$.

As a second step, we will define the notion of right and left derivative at a and b , respectively, at the end of this section.

The following lemma is needed for the constant function theorem:

Lemma 6. *Let J be an interval of $\bullet\mathbb{R}$. Then both the interval J and its interior $\text{int}(J)$ are connected in the Fermat topology.*

Proof. Let a, b be the endpoints of the interval J . We proceed for a, b finite and for the interval (a, b) , because the proof is similar for the other cases. Assume that there exist two non-empty relatively Fermat open sets U_1, U_2 of (a, b) such that $(a, b) = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Set $I := \{^\circ x \mid x \in (a, b)\}$. It is not hard to see that $I = [^\circ a, ^\circ b]_{\mathbb{R}} \neq \emptyset$. Moreover, setting $V_i := \{^\circ x \mid x \in U_i\}$ for $i = 1, 2$, we have that V_i is a non-empty relatively open set of I , $I = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, which implies the conclusion since I is connected in \mathbb{R} . The proof for the case of the interior is similar since $\text{int}(a, b) = \{x \in \bullet\mathbb{R} \mid ^\circ a < ^\circ x < ^\circ b\}$. \square

Here is the *constant function theorem*, which is a stepstone for the existence and uniqueness of primitives for smooth functions.

Theorem 7. *Let J be a non-infinitesimal interval of ${}^\bullet\mathbb{R}$. Let $f : J \rightarrow {}^\bullet\mathbb{R}$ be a smooth function such that $f'(x) = 0$ for each $x \in \text{int}(J)$. Then f is constant on J .*

Proof. Let a, b be the endpoints of the interval J . By the uniqueness Thm. 5, it is enough to show that f is a constant function when restricted to $\text{int}(J) = {}^\bullet\{({}^\circ a, {}^\circ b)_{\mathbb{R}}\}$, and the assumption that ${}^\circ a < {}^\circ b$ (non-infinitesimal interval) implies that $\text{int}(J) \neq \emptyset$. Let $x \in \text{int}(J)$, so that

$$\forall^0 h \in {}^\bullet\mathbb{R} : x + h \text{ is an interior point of } (a, b). \quad (10)$$

From the Fermat-Reyes Thm. 2 we get

$$\forall^0 h \in {}^\bullet\mathbb{R} : f(x + h) = f(x) + h \cdot f'[x, h]. \quad (11)$$

Let us say that (10) and (11) hold for $|h| < r \in \mathbb{R}_{>0}$ (here r depends on x), and fix such an h . Without loss of generality, we may assume that $|h_t| < r$ for each $t \geq 0$, where $(h_t)_t$ is a little-oh polynomial representing h .

For simplicity, set $y := f'[x, h]$. As proved in Thm. 2 (or Thm. 22 in [12]), we can always find an ordinary smooth function α and a parameter $p \in {}^\bullet\mathbb{R}^p$ such that for all $s \in [0, 1]_{\mathbb{R}}$

$$y_t = \int_0^1 \partial_2 \alpha(p_t, x_t + s \cdot h_t) \, ds \quad \forall^0 t \in \mathbb{R}_{\geq 0} \quad (12)$$

$$f'(x + sh) = {}^\bullet(\partial_2 \alpha)(p, x + sh) \text{ in } {}^\bullet\mathbb{R}, \quad (13)$$

where \int denotes the classical Riemann integral. As a motivation, we can say that we would like to consider the function $y = \int_0^1 {}^\bullet(\partial_2 \alpha)(p, x + sh) ds$ with $p, x, h \in {}^\bullet\mathbb{R}$. For fixed $p, x, h \in {}^\bullet\mathbb{R}$, we understand it as y_t above, i.e., using little-oh polynomial representative for each Fermat real, so that we get an ordinary integral. Note that this integral is independent of the representatives we choose. But $|sh| < r$, so that $x + sh$ is still an interior point, and hence by assumption $f'(x + s \cdot h) = 0$, i.e. ${}^\bullet(\partial_2 \alpha)(p, x + s \cdot h) = 0$ in ${}^\bullet\mathbb{R}$. Written in explicit form, this means

$$\lim_{t \rightarrow 0^+} \frac{\partial_2 \alpha(p_t, x_t + s \cdot h_t)}{t} = 0.$$

From (12), and using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{y_t}{t} &= \lim_{t \rightarrow 0^+} \frac{1}{t} \cdot \int_0^1 \partial_2 \alpha(p_t, x_t + s \cdot h_t) \, ds \\ &= \int_0^1 \lim_{t \rightarrow 0^+} \frac{\partial_2 \alpha(p_t, x_t + s \cdot h_t)}{t} \, ds = 0. \end{aligned}$$

So $y = f'[x, h] = 0$ in $\bullet\mathbb{R}$, and hence (11) yields $f(x + h) = f(x)$ for all h such that $|h| < r$. This proves that f is locally constant in the Fermat topology, and from Lem. 6 the conclusion follows. \square

We finally introduce the notion of left and right derivatives by means of an extension of the Fermat-Reyes theorem from a Fermat open set like $\text{int}(J)$ to the whole interval J . We first need the following:

Lemma 8. *Let $a, b \in \bullet\bar{\mathbb{R}}$ and let J be a non-infinitesimal interval of $\bullet\mathbb{R}$ with endpoints a, b . Let $f : J \rightarrow \bullet\mathbb{R}$ be a smooth function. Then there exist $\delta \in \mathbb{R}_{>0}$ and a smooth function $\bar{f} : (a - \delta, b + \delta) \rightarrow \bullet\mathbb{R}$ such that $\bar{f}|_J = f$.*

On the other hand, the smooth function $x \in \{x \in \bullet\mathbb{R} \mid 0 < {}^\circ x < 1\} \mapsto \frac{1}{x} \in \bullet\mathbb{R}$ cannot be extended to a smooth function defined on any interval containing an infinitesimal.

Proof. By definition, we can write the function f as a parameterized extension of an ordinary smooth function both in a Fermat open neighborhood of $a + dt$ and of $b - dt$, i.e.:

$$f(x) = \bullet\alpha(p, x) \quad \forall x \in \bullet A \cap J \quad (14)$$

$$f(x) = \bullet\beta(q, x) \quad \forall x \in \bullet B \cap J, \quad (15)$$

where A, B are open sets in \mathbb{R} such that ${}^\circ a \in A$ and ${}^\circ b \in B$, $\alpha \in \mathcal{C}^\infty(P \times A, \mathbb{R})$, $\beta \in \mathcal{C}^\infty(Q \times B, \mathbb{R})$, $p \in \bullet P \subseteq \bullet\mathbb{R}^p$, and $q \in \bullet Q \subseteq \bullet\mathbb{R}^q$. Therefore, any two of the smooth functions $\bullet\alpha(p, -) : \bullet A \rightarrow \mathbb{R}$, $f|_{\text{int}(J)}$ and $\bullet\beta(q, -) : \bullet B \rightarrow \mathbb{R}$ are equal in the intersection of their domains. Since ${}^\circ a \in A$ and ${}^\circ b \in B$, we can always find some $\delta \in \mathbb{R}_{>0}$ such that $\{\bullet A, \text{int}(J), \bullet B\}$ is a Fermat open covering of $(a - \delta, b + \delta)$ (let us note explicitly that in this step we need to use the trichotomy law of the total order \leq on $\bullet\mathbb{R}$). The sheaf property of the $\bullet\mathcal{C}^\infty$ space $(a - \delta, b + \delta)$ yields the existence of a smooth function $\bar{f} : (a - \delta, b + \delta) \rightarrow \mathbb{R}$ such that $\bar{f}|_J = f|_J$. \square

The notion of thickening \tilde{U}_v is defined only for Fermat open sets U ; see [12]. A simple way to extend this notion to arbitrary subsets is the following:

Definition 9. Let $C \subseteq \bullet\mathbb{R}^d$ and $v \in \bullet\mathbb{R}^d$. We say that $T \subseteq \bullet\mathbb{R}^d \times \bullet\mathbb{R}$ is a *thickening of C along v* if there exist a Fermat open set $U \subseteq \bullet\mathbb{R}^d$ such that $C \subseteq U$ and $T = \{(x, h) \in \tilde{U}_v \mid x \in C\}$.

Here is an extension of the Fermat-Reyes Thm. 2:

Lemma 10. *Let $a, b \in \bullet\bar{\mathbb{R}}$ and let J be a non-infinitesimal interval of $\bullet\mathbb{R}$ with endpoints a, b . Then the Fermat-Reyes theorem also holds if we replace the Fermat open set U with the interval J in the statement of Thm. 2. More*

precisely, if $x \in J$ and $x \simeq a$ (i.e., ${}^\circ x = {}^\circ a$), so that $x \in J \setminus \text{int}(J)$, then equation (6) shall be replaced by

$$\forall^0 h \in {}^\bullet\mathbb{R}_{>0} : f(x+h) = f(x) + h \cdot r(x, h) \quad \text{in } {}^\bullet\mathbb{R}.$$

Analogously, use $\forall^0 h \in {}^\bullet\mathbb{R}_{<0}$ if $x \simeq b$.

Remark 11. Let $r : T \longrightarrow {}^\bullet\mathbb{R}$ be an incremental ratio of a smooth function $f : J \longrightarrow {}^\bullet\mathbb{R}$ defined on a non-infinitesimal interval J of ${}^\bullet\mathbb{R}$, and let $s : S \longrightarrow {}^\bullet\mathbb{R}$ be an incremental ratio of the restriction $f|_{\text{int}(J)}$. Then s is essentially a restriction of r in the sense that

$$\forall x \in \text{int}(J) \forall^0 h \in {}^\bullet\mathbb{R} : r(x, h) = s(x, h).$$

We can hence define $f'(x) := r(x, 0)$ for all $x \in J$, obtaining an extension of f' from $\text{int}(J)$ to the whole J .

Proof. Let $\bar{f} : (a - \delta, b + \delta) \longrightarrow {}^\bullet\mathbb{R}$ be an extension of $f : J \longrightarrow {}^\bullet\mathbb{R}$ as in Lem. 8, set $U := \text{int}(a - \delta, b + \delta)$ and let $\bar{f}'[-, -] : \tilde{U} \longrightarrow {}^\bullet\mathbb{R}$ be the incremental ratio of \bar{f} . Then $T := \{(x, h) \in \tilde{U} \mid x \in J\}$ is a thickening of J and $r := \bar{f}'[-, -]|_T$ is a searched incremental ratio of f . The uniqueness part follows from the uniqueness Thm. 5. \square

5. Existence and uniqueness of primitives

We can now prove the existence and uniqueness of primitives for smooth functions defined on an interval $[a, b]$ with finite endpoints.

Theorem 12. *Let $a, b \in {}^\bullet\mathbb{R}$ with ${}^\circ a < {}^\circ b$, let $f : [a, b] \longrightarrow {}^\bullet\mathbb{R}$ be a smooth function, and let $u \in [a, b]$. Then there exists one and only one smooth function*

$$I : [a, b] \longrightarrow {}^\bullet\mathbb{R}$$

such that

$$I'(x) = f(x) \quad \forall x \in [a, b], \quad \text{and}$$

$$I(u) = 0.$$

Proof. We may assume that $u = a$, since if $I' = f$ on $[a, b]$ and $I(a) = 0$, then $J(x) := I(x) - I(u)$ verifies $J' = I' = f$ on $[a, b]$ and $J(u) = 0$.

For every $x \in [a, b]$, we can write

$$f|_{\mathcal{V}_x} = {}^\bullet\alpha_x(p_x, -)|_{\mathcal{V}_x}$$

for suitable $p_x \in {}^\bullet U_x \subseteq {}^\bullet\mathbb{R}^{\mathbb{P}^x}$, U_x an open subset of $\mathbb{R}^{\mathbb{P}^x}$, V_x an open subset of \mathbb{R} such that $x \in {}^\bullet V_x \cap [a, b] =: \mathcal{V}_x$ and $\alpha_x \in \mathcal{C}^\infty(U_x \times V_x, \mathbb{R})$. We may assume that the open sets V_x are of the form $V_x = ({}^\circ x - \delta_x, {}^\circ x + \delta_x)_\mathbb{R}$ for some $\delta_x \in \mathbb{R}_{>0}$.

The idea for the construction of the function I is to patch together suitable integrals of the functions $\bullet\alpha_x(p_x, -)$ and, at the same time, to respect the condition $I(a) = 0$. In order to get a quasi-standard smooth function, we have to patch together integrals in an order so that each one is the “continuation” of the previous one. In other words, on the non-empty connected intersection of the domains of any two of these integrals, the integrals must have the same value at one point, so that we can prove that they are equal on the whole intersection.

Since $(V_x)_{x \in [{}^\circ a, {}^\circ b]_{\mathbb{R}}}$ is an open cover of the compact set $[{}^\circ a, {}^\circ b]_{\mathbb{R}}$, we can find a finite subcover. That is, we can find $x_1, \dots, x_n \in [{}^\circ a, {}^\circ b]_{\mathbb{R}}$ such that $(V_{x_i})_{i=1, \dots, n}$ is an open cover of $[{}^\circ a, {}^\circ b]_{\mathbb{R}}$. We will use simplified notations like $V_i := V_{x_i}$, $\delta_i := \delta_{x_i}$, $\alpha_i := \alpha_{x_i}$, etc.

By shrinking each V_i if necessary, we may always assume to have chosen the indices $i = 1, \dots, n$ and the radii $\delta_i \in \mathbb{R}_{>0}$ such that

$${}^\circ a = x_1 < x_2 < \dots < x_n = {}^\circ b, \text{ and}$$

$$x_i - \delta_i < x_{i+1} - \delta_{i+1} < x_i + \delta_i < x_{i+1} + \delta_{i+1} \quad \forall i = 1, \dots, n-1.$$

In this way, the intervals $V_i = (x_i - \delta_i, x_i + \delta_i)_{\mathbb{R}}$ and $V_{i+1} = (x_{i+1} - \delta_{i+1}, x_{i+1} + \delta_{i+1})_{\mathbb{R}}$ intersect in $(x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}$. For each $i = 1, \dots, n-1$, we choose a point $\bar{x}_i \in (x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}$ and define, recursively:

$$\begin{aligned} \iota_1(q, x, y) &:= \int_y^x \alpha_1(q, s) ds \quad \forall q \in U_{x_1} \forall x, y \in V_1, \\ I_1(x) &:= \bullet\iota_1(p_1, x, a) \quad \forall x \in \bullet V_1. \end{aligned} \quad (16)$$

$$\begin{aligned} \iota_{i+1}(q, x) &:= \int_{\bar{x}_i}^x \alpha_{i+1}(q, s) ds \quad \forall q \in U_{x_i} \forall x \in V_{i+1} \forall i = 1, \dots, n-1, \\ I_{i+1}(x) &:= \bullet\iota_{i+1}(p_{i+1}, x) + I_i(\bar{x}_i) \quad \forall x \in \bullet V_{i+1} \forall i = 1, \dots, n-1. \end{aligned} \quad (17)$$

So every I_i is a quasi-standard smooth function defined on $\bullet V_i$, and moreover, from Thm. 2 it follows that

$$I'_i(x) = \bullet\alpha_i(p_i, x) = f(x) \quad \forall x \in \bullet V_i.$$

Therefore, for each point x in

$$\begin{aligned} \bullet V_i \cap \bullet V_{i+1} &= \bullet(V_i \cap V_{i+1}) = \bullet\{(x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}\} \\ &= \text{int}([x_{i+1} - \delta_{i+1}, x_i + \delta_i]), \end{aligned}$$

we have

$$\begin{aligned} I'_i(x) &= f(x) = I'_{i+1}(x), \text{ and} \\ (I_i - I_{i+1})(\bar{x}_i) &= 0. \end{aligned} \quad (18)$$

So, from the constant function Thm. 7 it follows that $I_i = I_{i+1}$ on $[x_{i+1} - \delta_{i+1}, x_i + \delta_i]$. We can hence use the Fermat open cover $(\bullet V_i \cap [a, b])_{i=1, \dots, n}$ of the space $[a, b] \in \bullet \mathcal{C}^\infty$ to patch together the functions $I_i|_{\bullet V_i \cap [a, b]}$ to obtain a smooth function $I : [a, b] \rightarrow \bullet \mathbb{R}$, by the sheaf property of smooth functions. This function satisfies the required conditions because of (18) and the equalities $I(a) = I_1(a) = 0$.

To prove the uniqueness, suppose that J verifies $J' = f$ on $[a, b]$ and $J(u) = 0$. Then again by the constant function Thm. 7, we know that $(J - I)|_{[a, b]}$ is constant and equal to zero at u , and hence a zero function. \square

The above theorem can be extended to smooth functions defined on an arbitrary non-infinitesimal interval.

Theorem 13. *Let $a, b \in \bullet \bar{\mathbb{R}}$ and let J be a non-infinitesimal interval of $\bullet \mathbb{R}$ with endpoints a, b . Let $f : J \rightarrow \bullet \mathbb{R}$ be a smooth function, and let $u \in J$. Then there exists one and only one smooth function $I : J \rightarrow \bullet \mathbb{R}$ such that $I' = f$ and $I(u) = 0$.*

Proof. We proceed for the interval $(-\infty, +\infty)$, since the other cases are similar. Set $r := {}^\circ u$, and for each $k \in \mathbb{N}_{>0}$, define

$$f_k := f|_{[r-k, r+k]}.$$

By Thm. 12, for each k there exists a unique smooth function $I_k : [r-k, r+k] \rightarrow \bullet \mathbb{R}$ such that $I'_k(x) = f_k(x) = f(x)$ for $x \in \text{int}[r-k, r+k] =: V_k$ and $I_k(u) = 0$. Note that $(V_k)_{k>0}$ is a Fermat open cover of $\bullet \mathbb{R}$. Moreover, since $I'_k(x) = f(x) = I'_j(x)$ for every $x \in V_k \cap V_j$ and $I_k(u) = I_j(u)$, by the constant function Thm. 7, I_k and I_j coincide on $V_k \cap V_j$. By the sheaf property of smooth functions, we get

$$\exists! I \in \bullet \mathcal{C}^\infty(\bullet \mathbb{R}, \bullet \mathbb{R}) : I|_{V_k} = I_k \quad \forall k \in \mathbb{N}_{>0}. \quad (19)$$

Note that for any $x \in \bullet \mathbb{R}$, there exists $k = k(x) \in \mathbb{N}_{>0}$ such that $x \in \text{int}(V_k)$. Therefore, for each $x \in \bullet \mathbb{R}$ and for $h \in \bullet \mathbb{R}$ sufficiently small, we have

$$I(x+h) = I_k(x+h) = I(x) + h \cdot I'[x, h] = I_k(x) + h \cdot I'_k[x, h].$$

This and the uniqueness Thm. 5 imply that the incremental ratios of I and I_k are equal, i.e., $I'[x, h] = I'_k[x, h]$. Thus, $I'(x) = I'_k(x) = f(x)$, and finally $I(u) = I_1(u) = 0$.

This proves the existence part.

Since the restriction of a primitive of f to $[r-k, r+k]$ is a primitive of f_k for each k , the uniqueness then follows from Thm. 12. \square

We can now define

Definition 14. Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$. Let $f : J \rightarrow \bullet\mathbb{R}$ be a smooth function, and let $u \in J$. We define $\int_u^{(-)} f$ to be the unique function verifying the following properties:

- (i) $\int_u^{(-)} f := \int_u^{(-)} f(s) \, ds : J \rightarrow \bullet\mathbb{R}$ is a smooth function;
- (ii) $\int_u^u f = 0$;
- (iii) $\forall x \in J : \left(\int_u^{(-)} f \right)'(x) = \frac{d}{dx} \int_u^x f(s) \, ds = f(x)$.

Remark 15.

1. Note that, although we require the interval J to be non-infinitesimal for the domain of the smooth function f , the integral $\int_u^v f$ is also defined when $v \in J$ and $v \simeq u$.
2. It is not hard to see that if $f, g : J \rightarrow \bullet\mathbb{R}$ are smooth functions, $a, b \in J$ with $a < b$, and $f|_{(a,b)} = g|_{(a,b)}$, then $\int_a^b f = \int_a^b g$. We get the same conclusion if ${}^\circ a < {}^\circ b$ and we assume only $f|_{\text{int}(a,b)} = g|_{\text{int}(a,b)}$.
3. From Def. 14, we obtain a generalization of the usual notion of integral. Indeed, for $a, b, u \in \mathbb{R}$ with $a < u < b$, let $f \in \mathcal{C}^\infty([a, b]_\mathbb{R}, \mathbb{R})$. We extend f to an ordinary smooth function defined on an open interval $(a - \delta, b + \delta)_\mathbb{R}$, for some $\delta \in \mathbb{R}_{>0}$. We still use the symbol f to denote this extension. We consider the quasi-standard smooth function

$$I := \bullet \left(\int_u^{(-)} f(s) \, ds \right) : \bullet(a - \delta, b + \delta)_\mathbb{R} \rightarrow \bullet\mathbb{R}.$$

Now, we have that

$$[a, b] \subseteq \bullet(a - \delta, b + \delta)_\mathbb{R}$$

so that we can consider the restriction $I|_{[a,b]}$. It is not hard to prove that this restriction verifies all the properties of the previous Def. 14 for the function $\bullet f$. Meanwhile, because I is the extension of an ordinary smooth function, it also verifies

$$\forall x \in [a, b]_\mathbb{R} : \int_u^x \bullet f(s) \, ds = I(x) = \bullet \left(\int_u^x f(s) \, ds \right) = \int_u^x f(s) \, ds \in \mathbb{R}. \quad (20)$$

The recursive definitions (16) and (17) applied to $f|_{[u,v]}$ yield the following useful result:

Lemma 16. Let $a, b \in {}^\bullet\bar{\mathbb{R}}$ and let J be a non-infinitesimal interval of ${}^\bullet\mathbb{R}$ with endpoints a, b . Let $f : J \rightarrow {}^\bullet\mathbb{R}$ be a smooth function, and let $u, v \in J$ with $u \leq v$. Then there exist $n \in \mathbb{N}_{>0}$, $y, \bar{y}, \delta \in \mathbb{R}^n$, $(\alpha_i)_{i=1}^n, (p_i)_{i=1}^n$ such that

- (i) $\alpha_i \in C^\infty(U_i \times V_i, \mathbb{R})$ and $p_i \in {}^\bullet U_i$, where $U_i \subseteq \mathbb{R}^{p_i}$, $V_i = (y_i - \delta_i, y_i + \delta_i)_{\mathbb{R}}$, $\delta_i \in \mathbb{R}_{>0}$, $\forall i = 1, \dots, n$
- (ii) $f(x) = {}^\bullet\alpha_i(p_i, x) \quad \forall x \in {}^\bullet V_i \cap J \quad \forall i = 1, \dots, n$
- (iii) $\bar{y}_i \in (y_{i+1} - \delta_{i+1}, y_i + \delta_i)_{\mathbb{R}} = V_i \cap V_{i+1} \neq \emptyset \quad \forall i = 1, \dots, n-1$
- (iv) ${}^\circ u = y_1 < y_2 < \dots < y_n = {}^\circ v$ if ${}^\circ u < {}^\circ v$ and ${}^\circ u = x_1 = {}^\circ v$ otherwise
- (v) $\int_u^v f = \int_u^{\bar{y}_1} {}^\bullet\alpha_1(p_1, s) ds + \int_{\bar{y}_1}^{\bar{y}_2} {}^\bullet\alpha_2(p_2, s) ds + \dots + \int_{\bar{y}_{n-1}}^v {}^\bullet\alpha_n(p_n, s) ds$, and a little-oh polynomial representing $\int_u^v f$ is given by

$$t \in \mathbb{R}_{\geq 0} \mapsto \int_{u_t}^{\bar{y}_1} \alpha_1((p_1)_t, s) ds + \dots + \int_{\bar{y}_{n-1}}^{v_t} \alpha_n((p_n)_t, s) ds \in \mathbb{R}.$$

6. Classical formulas of integral calculus

In this section and the first part of next, we will prove the classical formulas of integral calculus.

The first results concerning the integral calculus are the following:

Theorem 17. Let J be a non-infinitesimal interval of ${}^\bullet\mathbb{R}$, let $f, g : J \rightarrow {}^\bullet\mathbb{R}$ be smooth functions, and let $u, v \in J$. Then we have:

- (i) $\int_u^v (f + g) = \int_u^v f + \int_u^v g$
- (ii) $\int_u^v \lambda f = \lambda \int_u^v f \quad \forall \lambda \in {}^\bullet\mathbb{R}$
- (iii) $\int_u^v f' = f(v) - f(u)$
- (iv) $\int_u^v (f' \cdot g) = [f \cdot g]_u^v - \int_u^v (f \cdot g')$

We will prove the additivity with respect to the integration intervals, i.e., $\int_u^v f + \int_v^w f = \int_u^w f$, after the proof of the commutativity of differentiation and integration in next section. The latter relies on the quasi-standard smoothness of the function $x \mapsto \int_0^1 f(s, x) ds$, which will be proved as a consequence of the ${}^\bullet C^\infty$ smoothness of suitable infinite dimensional integral operators.

Remark 18. In fact, one can prove the above properties together with the additivity with respect to the integration intervals using Lem. 16. As an example, we prove in this remark that $\int_u^v f + \int_v^w f = \int_u^w f$ for any smooth function f defined on a non-infinitesimal interval J of ${}^\bullet\mathbb{R}$ with $u, v, w \in J$. By Lem. 16, we can write

$$\begin{aligned}\int_u^v f &= \int_u^{\bar{y}_1} \bullet \alpha_1(p_1, s) ds + \dots + \int_{\bar{y}_{n-1}}^v \bullet \alpha_n(p_n, s) ds \\ \int_v^w f &= \int_v^{\bar{z}_1} \bullet \beta_1(q_1, s) ds + \dots + \int_{\bar{z}_{m-1}}^w \bullet \beta_m(q_m, s) ds.\end{aligned}$$

Since the function f is quasi-standard smooth, we may assume that $\alpha_n = \beta_1$, i.e., $\alpha_n(p_1, x) = f(x) = \beta_1(q_1, x)$ for every $x \in (\bar{y}_{n-1}, \bar{z}_1)$. The conclusion then follows from the additivity of ordinary integral with respect to ordinary integration intervals.

This proof uses local expressions of quasi-standard smooth functions. The proofs given below are of “global” nature, which is the point we try to emphasize in the present work.

Proof. In the following, we will use the notations

$$F := \int_u^{(-)} f : J \longrightarrow \bullet \mathbb{R} \quad , \quad G := \int_u^{(-)} g : J \longrightarrow \bullet \mathbb{R}.$$

To prove (i), it suffices to note that $F + G$ is smooth and

$$\begin{aligned}(F + G)(u) &= F(u) + G(u) = 0, \\ (F + G)'(x) &= F'(x) + G'(x) = f(x) + g(x) = (f + g)(x) \quad \forall x \in J.\end{aligned}$$

From the uniqueness of Def. 14, the conclusion $F + G = \int_u^{(-)} (f + g)$ follows.

For the proofs of the other properties, we can follow the same method: we define a suitable quasi-standard smooth function H starting from the right hand side of the equality we need to prove, and we prove that H verifies the same properties that characterize uniquely the function on the left hand side. For example, to prove (ii) we consider $H(x) := \lambda F(x)$; to prove (iii) we define $H(x) := f(x) - f(u)$; and finally to prove (iv) we set $H(x) := f(x) \cdot g(x) - f(u) \cdot g(u) - \int_u^x (f \cdot g')$. \square

Here is the result of *integration by substitution*:

Theorem 19. *Let J, J_1 be non-infinitesimal intervals of $\bullet \mathbb{R}$, let $u, v \in J_1$, and let*

$$J_1 \xrightarrow{\varphi} J \xrightarrow{f} \bullet \mathbb{R}$$

be smooth functions. Then

$$\int_{\varphi(u)}^{\varphi(v)} f(t) dt = \int_u^v f[\varphi(s)] \cdot \varphi'(s) ds.$$

Proof. Define

$$\begin{aligned} F(x) &:= \int_{\varphi(u)}^x f \quad \forall x \in J \\ H(y) &:= \int_{\varphi(u)}^{\varphi(y)} f \quad \forall y \in J_1 \\ G(y) &:= \int_u^y f[\varphi(s)] \cdot \varphi'(s) \, ds \quad \forall y \in J_1. \end{aligned}$$

Each one of these functions is smooth because of Def. 14 (we recall that $u \in J_1$ is fixed) or because it can be written as composition of smooth functions. We have $H(u) = G(u) = 0$, $H(y) = F[\varphi(y)]$ for every $y \in J_1$, and by the chain rule ([12, Thm. 29]) $H'(y) = F'[\varphi(y)] \cdot \varphi'(y) = f[\varphi(y)] \cdot \varphi'(y) = G'(y)$ for each $y \in J_1$. From the uniqueness of Thm. 12 the conclusion $H = G$ follows. \square

7. Infinite dimensional integral operators

If we want to prove, in our framework, that

$$\frac{d}{dx} \left(\int_0^1 f(s, x) \, ds \right) = \int_0^1 \frac{\partial f}{\partial x}(s, x) \, ds, \quad (21)$$

we need to prove that the function defined by

$$x \in [a, b] \mapsto \int_0^1 f(s, x) \, ds \in \bullet\mathbb{R} \quad (22)$$

is smooth. Of course, in (21) $\frac{\partial f}{\partial x}(s, x) := [f(s, -)]'(x)$. The smoothness of (22) follows as a trivial consequence of the smoothness of the function defined by

$$I_a^b : (f, u, v) \in \bullet\mathcal{C}^\infty([a, b], \bullet\mathbb{R}) \times [a, b]^2 \mapsto \int_u^v f \in \bullet\mathbb{R}. \quad (23)$$

In fact, the function defined in (22) can be written as $I_0^1(-, 0, 1) \circ (f \circ \tau)^\wedge$, where⁵

$$\tau : (x, s) \in [a, b] \times [0, 1] \mapsto (s, x) \in [0, 1] \times [a, b].$$

Therefore, the smoothness of (22) follows from the smoothness of (23) and the Cartesian closedness of the category $\bullet\mathcal{C}^\infty$ of Fermat spaces. For simplicity, in the following we will denote by $\bullet\mathbb{R}^{[a, b]}$ the space $\bullet\mathcal{C}^\infty([a, b], \bullet\mathbb{R})$.

⁵See [12, Def. 1] for the notations $(-)^^\wedge$ and $(-)^^\vee$.

From Thm. 19, we have

$$\int_u^v f = \int_0^1 f[y \cdot (v - u) + u] \cdot (v - u) dy.$$

Thereby, setting

$$\begin{aligned} \psi(f, u, v) : y \in [0, 1] &\mapsto f[y \cdot (v - u) + u] \cdot (v - u) \in \bullet\mathbb{R} \\ I : f \in \bullet\mathbb{R}^{[0,1]} &\mapsto \int_0^1 f \in \bullet\mathbb{R}, \end{aligned}$$

it is easy to show that

$$\psi : \bullet\mathbb{R}^{[a,b]} \times [a, b]^2 \longrightarrow \bullet\mathbb{R}^{[0,1]} \text{ is smooth}$$

and $I_a^b(f, u, v) = I[\psi(f, u, v)]$. Therefore, the following result suffices to prove that the function I_a^b is smooth.

Lemma 20. *The function*

$$I : f \in \bullet\mathbb{R}^{[0,1]} \mapsto \int_0^1 f \in \bullet\mathbb{R}$$

is smooth, i.e., it is an arrow of the category $\bullet\mathcal{C}^\infty$.

Proof. Take a figure of $\bullet\mathbb{R}^{[0,1]}$, i.e., let

$$\begin{aligned} S &\subseteq \bullet\mathbb{R}^s \\ d &\in \bullet\mathcal{C}^\infty(S, \bullet\mathbb{R}^{[0,1]}). \end{aligned} \tag{24}$$

Recall that here we use the simplified symbol $[0, 1]$ to denote the corresponding subspace of $\bullet\mathbb{R}$, i.e., $([0, 1] \prec \bullet\mathbb{R})$. From Property 5(i) of Thm. 18 in [12] we have that $\overline{[0, 1]} = ([0, 1] \prec \bullet\mathbb{R}) = \overline{[0, 1]}$. Therefore, from (24), we know that $d^\vee : \bar{S} \times \overline{[0, 1]} \longrightarrow \bullet\mathbb{R}$ is smooth. We need to prove that the function

$$s \in S \mapsto \int_0^1 d(s) = \int_0^1 d^\vee(s, u) du \in \bullet\mathbb{R} \tag{25}$$

is smooth, i.e., that it belongs to $\mathbf{S}\bullet\mathbb{R}^\infty(S, \bullet\mathbb{R})$. From Properties 5(h) and 5(i) of Thm. 18 in [12], we have that

$$\begin{aligned} \bar{S} \times \overline{[0, 1]} &= (S \prec \bullet\mathbb{R}^s) \times ([0, 1] \prec \bullet\mathbb{R}) \\ &= ((S \times [0, 1]) \prec \bullet\mathbb{R}^{s+1}) \\ &= \overline{S \times [0, 1]}. \end{aligned}$$

Hence, $d^\vee : \overline{S \times [0, 1]} \longrightarrow \bullet\mathbb{R}$ is smooth, i.e.,

$$d^\vee : S \times [0, 1] \longrightarrow \bullet\mathbb{R} \text{ is an arrow of } \mathbf{S}\bullet\mathbb{R}^\infty, \tag{26}$$

because $\mathbf{S}^\bullet \mathbb{R}^\infty$ is fully embedded in the category $\bullet \mathcal{C}^\infty$ of Fermat spaces (see [12, Thm. 18(4)]). Property (26), for every $(s, u) \in S \times [0, 1]$, yields the existence of

$$p_{s,u} \in \bullet \mathbb{R}^{\mathbf{p}_{s,u}}$$

$\mathcal{U}_{s,u}$ Fermat open neighbourhood of $p_{s,u}$ defined by $U_{s,u}$

$\mathcal{V}_{s,u}$ Fermat open neighbourhood of (s, u) defined by $V_{s,u}$ in $S \times [0, 1]$

$$\alpha_{s,u} \in \mathcal{C}^\infty(U_{s,u} \times V_{s,u}, \mathbb{R})$$

such that $d^\vee|_{\mathcal{V}_{s,u}} = \bullet \alpha_{s,u}(p_{s,u}, -)|_{\mathcal{V}_{s,u}}$. Now we fix $s \in S$, and everything above only depends on $u \in [0, 1]$. Since $(s, u) \in \mathcal{V}_u = \bullet V_u \cap (S \times [0, 1])$ and V_u is open in \mathbb{R}^{s+1} , we have that $(\circ s, \circ u) \in V_u$. Thus, we may always assume that $V_u = A_u \times B_u$, for some open subsets A_u in \mathbb{R}^s and B_u in \mathbb{R} , with $(\circ s, \circ u) \in A_u \times B_u$. Hence, $(s, u) \in \bullet (A_u \times B_u) = \bullet A_u \times \bullet B_u = \bullet V_u$. In this way, we obtain an open cover $(B_u)_{u \in [0, 1]_\mathbb{R}}$ of $[0, 1]_\mathbb{R}$, and from it we can extract a finite subcover. Therefore, we can find $u_1, \dots, u_n \in [0, 1]_\mathbb{R}$ such that $(B_{u_i})_{i=1}^n$ is an open cover of $[0, 1]_\mathbb{R}$. As in Thm. 12, we may assume to have chosen the indices $i = 1, \dots, n$ and have found suitable δ_i 's in $\mathbb{R}_{>0}$ so that

$$0 = u_1 < u_2 < \dots < u_n = 1$$

$$B_{u_i} = (u_i - \delta_i, u_i + \delta_i)_\mathbb{R}$$

$$B_{u_i} \cap B_{u_{i+1}} = (u_{i+1} - \delta_{i+1}, u_i + \delta_i)_\mathbb{R} \neq \emptyset \quad \forall i = 1, \dots, n-1.$$

Finally, set $A := A_{u_1} \cap \dots \cap A_{u_n}$, which is an open neighbourhood of $\circ s$, so that $s \in \bullet A$. Take a point $\bar{u}_i \in (u_{i+1} - \delta_{i+1}, u_i + \delta_i)_\mathbb{R}$. For every $x \in \bullet A \cap S$, we have

$$d^\vee(x, u) = \bullet \alpha_{u_i}(p_{u_i}, x, u) \quad \forall u \in \bullet B_{u_i} \cap [0, 1],$$

and hence Lem. 16 implies that

$$\begin{aligned} \int_0^1 d^\vee(x, u) du &= \int_0^{\bar{u}_1} \bullet \alpha_{u_1}(p_{u_1}, x, s) ds + \int_{\bar{u}_1}^{\bar{u}_2} \bullet \alpha_{u_2}(p_{u_2}, x, s) ds + \dots \\ &\quad + \int_{\bar{u}_{n-1}}^1 \bullet \alpha_{u_n}(p_{u_n}, x, s) ds =: \bullet \gamma(\bar{p}, x), \end{aligned}$$

where $\bar{p} := (p_{u_1}, \dots, p_{u_n}) \in \mathcal{U}_{u_1} \times \dots \times \mathcal{U}_{u_n} \subseteq \bullet \mathbb{R}^{\mathbf{p}_{u_1} + \dots + \mathbf{p}_{u_n}}$. From the classical version of the differentiation under the integral sign, we have that $\gamma \in \mathcal{C}^\infty(U \times A, \mathbb{R})$, where $U := U_{u_1} \times \dots \times U_{u_n}$, and hence we get the conclusion. \square

Corollary 21. *Let J be a non-infinitesimal interval of $\bullet \mathbb{R}$. Then the function*

$$(f, u, v) \in \bullet \mathcal{C}^\infty(J, \bullet \mathbb{R}) \times J^2 \mapsto \int_u^v f \in \bullet \mathbb{R}$$

is smooth.

We can now prove the differentiation under the integral sign:

Lemma 22. *Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$, and let $f : [0, 1] \times J \longrightarrow \bullet\mathbb{R}$ be a smooth function. Then*

$$\frac{d}{dx} \left(\int_0^1 f(s, x) ds \right) = \int_0^1 \frac{\partial f}{\partial x}(s, x) ds \quad \forall x \in J.$$

Proof. From Cartesian closedness, it follows that the map $x \in J \mapsto f(-, x) \in \bullet\mathbb{R}^{[0,1]}$ is smooth. Together with Lem. 20, we know that

$$x \in J \mapsto \int_0^1 f(s, x) ds \in \bullet\mathbb{R}$$

is smooth. Let $\partial_2 f[-, -, -]$ be the incremental ratio of f with respect to its second variable. Then for each $x \in \text{int}(J)$ and for $h \in \bullet\mathbb{R}$ sufficiently small, we have

$$\begin{aligned} \int_0^1 f(s, x + h) ds &= \int_0^1 \{f(s, x) + h \cdot \partial_2 f[s, x; h]\} ds \\ &= \int_0^1 f(s, x) ds + h \cdot \int_0^1 \partial_2 f[s, x; h] ds. \end{aligned}$$

Therefore, the uniqueness of the incremental ratio $R[-, -]$ of the function $\int_0^1 f(s, -) ds$ yields that

$$R[x, h] = \int_0^1 \partial_2 f[s, x; h] ds \quad \forall^0 h.$$

Hence,

$$\begin{aligned} \frac{d}{dx} \left(\int_0^1 f(s, x) ds \right) &= R[x, 0] \\ &= \int_0^1 \partial_2 f[s, x; 0] ds \\ &= \int_0^1 \frac{\partial f}{\partial x}(s, x) ds, \end{aligned}$$

as expected. □

The following lemma is needed for the “global” proof of the additivity of integral with respect to the integration intervals:

Lemma 23. *Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$, let $f : J \longrightarrow \bullet\mathbb{R}$ be a smooth function, and let $v \in J$. Then*

$$\frac{d}{dx} \left(\int_x^v f \right) = -f(x).$$

Proof. Set $\varphi(s) := x + (v - x)s$ for each $s \in [0, 1]$. From Thm. 19 (integration by substitution), we get

$$\int_x^v f(t) dt = \int_0^1 f[x + (v - x)s] \cdot (v - x) ds.$$

From this, using Lem. 22 (differentiation under the integral sign), the product rule ([12, Thm. 28]) and the chain rule ([12, Thm. 29]), we get

$$\begin{aligned} \frac{d}{dx} \left(\int_x^v f \right) &= - \int_0^1 f[x + (v - x)s] ds + \\ &\quad + (v - x) \int_0^1 f'[x + (v - x)s] \cdot (1 - s) ds \\ &= - \int_0^1 f[x + (v - x)s] ds + \\ &\quad + \int_x^v f'(t) dt - \int_0^1 s \cdot f'[\varphi(s)] \cdot \varphi'(s) ds. \end{aligned} \quad (27)$$

Integration by parts (Thm. 17(iv)) applied to the last integral yields

$$\begin{aligned} \int_0^1 s \cdot f'[\varphi(s)] \cdot \varphi'(s) ds &= [(f \circ \varphi)(s) \cdot s]_0^1 - \int_0^1 (f \circ \varphi)(s) ds \\ &= f(v) - \int_0^1 f[x + (v - x)s] ds. \end{aligned}$$

Substituting in (27) and using Thm. 17(iii), we obtain

$$\begin{aligned} \frac{d}{dx} \left(\int_x^v f \right) &= - \int_0^1 f[x + (v - x)s] ds + f(v) - f(x) - f(v) + \\ &\quad + \int_0^1 f[x + (v - x)s] ds \\ &= -f(x), \end{aligned}$$

as desired. \square

Now we are led to the “global” proof of the additivity of integral with respect to the integration intervals:

Corollary 24. *Let J be a non-infinitesimal interval of ${}^\bullet\mathbb{R}$, let $f : J \longrightarrow {}^\bullet\mathbb{R}$ be a smooth function, and let $u, v, w \in J$. Then we have:*

$$\int_u^v f + \int_v^w f = \int_u^w f.$$

Proof. Set $F(x) := \int_u^x f + \int_x^w f$ for all $x \in J$. Then Lem. 23 yields that $F'(x) = f(x) - f(x) = 0$ at each $x \in J$. Hence by Lem. 6, F is constant and $F(v) = F(u)$, which is our conclusion since $\int_u^u f = 0$. \square

Using Cor. 24, we can calculate the incremental ratio of the integral function $\int_u^{(-)} f$ and get a form of mean value theorem called Hadamard's lemma as follows:

Corollary 25. *Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$, let $f : J \rightarrow \bullet\mathbb{R}$ be a smooth function, and let $u \in J$. Then*

- (i) *for each $x \in J$ and $h \in \bullet\mathbb{R}$, if $x + h \in J$, then $\int_x^{x+h} f(s) ds = h \cdot \int_0^1 f(x + hs) ds$. Therefore, the incremental ratio of the smooth function $\int_u^{(-)} f$ is*

$$\left(\int_u^{(-)} f \right)' [x, h] = \int_0^1 f(x + hs) ds \quad \forall x \in \text{int}(J) \forall^0 h \in \bullet\mathbb{R};$$

- (ii) $\forall x \in \text{int}(J) \forall^0 h \in \bullet\mathbb{R} : f(x + h) - f(x) = h \cdot \int_0^1 f'(x + hs) ds$.

Proof. (i): The first statement follows from Thm. 19, and the second statement then follows from Cor. 24 and Thm. 2.

(ii): It follows by applying (i) to the smooth function $f' : \text{int}(J) \rightarrow \bullet\mathbb{R}$ together with Thm. 17(iii). \square

Standard and infinitesimal parts of an integral

To find a useful formula for the standard and the infinitesimal parts of an integral, we introduce the following:

Definition 26. Let $S \subseteq \bullet\mathbb{R}^n$ and let $f : S \rightarrow \bullet\mathbb{R}^d$ be a smooth function. Then

- (i) ${}^\circ S := \{ {}^\circ x \in \mathbb{R}^n \mid x \in S \}$.
- (ii) If ${}^\circ S \subseteq S$, then we set ${}^\circ f : r \in {}^\circ S \mapsto {}^\circ [f(r)] \in \mathbb{R}^d$, which is called the *standard part* of f .
- (iii) Assume that ${}^\circ S$ is an open subset of \mathbb{R}^n , and $\bullet({}^\circ S) = S$. Then we set ${}^\sigma f := \bullet({}^\circ f)$ and $\delta f := f - {}^\sigma f$, which are called, respectively, the *shadow* and the *infinitesimal* part of the smooth function f . Analogously, we set $\delta x := x - {}^\circ x$ for any $x \in \bullet\mathbb{R}$. Note that ${}^\sigma f(r) = {}^\circ f(r)$ for each $r \in {}^\circ S$.

For example, if $S := \text{int}(a, b)$, then ${}^\circ S = ({}^\circ a, {}^\circ b)_\mathbb{R} \subseteq S$, ${}^\circ S$ is an open subset of \mathbb{R} , and $\bullet({}^\circ S) = S$. Note that the standard part ${}^\circ f$ is a function defined and with values in standard points, whereas the shadow ${}^\sigma f$ can have nonstandard values for nonstandard points in the domain. Therefore, $(\delta f)(x)$ is not necessarily equal to $\delta(f(x))$, unless $x \in {}^\circ S \subseteq S$; see (i) of the following lemma. The correctness of the definition of shadow is proved in (iii) of the following lemma:

Lemma 27. Let $S \subseteq \bullet \mathbb{R}^n$. Assume that ${}^\circ S \subseteq S$, and let $f : S \longrightarrow \bullet \mathbb{R}^d$ be a smooth function. Then

- (i) $({}^\circ f)({}^\circ x) = {}^\circ [f(x)] \quad \forall x \in S$.
- (ii) If ${}^\circ S$ is an open subset of \mathbb{R}^n , then ${}^\circ f \in \mathcal{C}^\infty({}^\circ S, \mathbb{R}^d)$.
- (iii) If ${}^\circ S$ is an open subset of \mathbb{R}^n and $\bullet({}^\circ S) = S$, then ${}^\sigma f \in \bullet \mathcal{C}^\infty(S, \bullet \mathbb{R}^d)$ and $\delta f \in \bullet \mathcal{C}^\infty(S, D_\infty^d)$.
- (iv) Under the assumptions of (iii), if we further assume that $n = 1$, $S \supseteq \text{int}(a, b)$ and $u, v \in \text{int}(a, b)$, then the standard part of the integral $\int_u^v f$ is given by

$${}^\circ \left(\int_u^v f \right) = \int_{{}^\circ u}^{{}^\circ v} {}^\circ f = \int_{{}^\circ u}^{{}^\circ v} {}^\sigma f,$$

and the infinitesimal part is given by

$$\begin{aligned} \delta \left(\int_u^v f \right) &= \int_u^{{}^\circ u} {}^\sigma f + \int_{{}^\circ v}^v {}^\sigma f + \int_u^v \delta f \\ &= \sum_{i=1}^{[\omega(v)]} \frac{{}^\sigma f^{(i-1)}({}^\circ v)}{i!} (\delta v)^i - \sum_{j=1}^{[\omega(u)]} \frac{{}^\sigma f^{(j-1)}({}^\circ u)}{j!} (\delta u)^j + \int_u^v \delta f \\ &\in D_\infty, \end{aligned}$$

where $[\omega(u)]$ denotes the integer part of $\omega(u)$.

As trivial consequences of (iv), any integral over an infinitesimal integration interval is infinitesimal, and any integral of an infinitesimal-valued smooth function is infinitesimal.

Proof. (i): We can write $f(y) = \bullet \alpha(p, y)$ for each $y \in \bullet V \cap S$, where ${}^\circ x \in V$ and V is an open subset of \mathbb{R}^n . Therefore, $({}^\circ f)({}^\circ x) = {}^\circ [f({}^\circ x)] = {}^\circ [\bullet \alpha(p, {}^\circ x)] = \alpha({}^\circ p, {}^\circ x)$. On the other side, ${}^\circ [f(x)] = {}^\circ [\bullet \alpha(p, x)] = \alpha({}^\circ p, {}^\circ x)$.

(ii): We write again $f|_{\bullet V \cap S} = \bullet \alpha(p, -)|_{\bullet V \cap S}$, where now V is an open neighborhood of $r \in {}^\circ S \subseteq S$. But $V \cap {}^\circ S \subseteq \bullet V \cap S$, so ${}^\circ f|_{V \cap {}^\circ S} = \alpha({}^\circ p, -)|_{V \cap {}^\circ S}$, $V \cap {}^\circ S$ is an open neighborhood of r and $\alpha({}^\circ p, -)|_{V \cap {}^\circ S}$ is a smooth function. This proves our claim.

(iii): This is clear from the property of the extension $\bullet(-)$ of ordinary smooth functions and from the assumption $\bullet({}^\circ S) = S$. Observe that, using the previous notations, ${}^\sigma f(x) = \bullet({}^\circ f)(x) = \bullet \alpha({}^\circ p, x)$ for $x \in S$ (with α and p depending on x). Hence, $(\delta f)(x) = f(x) - {}^\sigma f(x) = \bullet \alpha(p, x) - \bullet \alpha({}^\circ p, x) \in D_\infty^d$.

(iv): Lem. 16 yields

$${}^\circ \left(\int_u^v f \right) = {}^\circ \left[\int_u^{\bar{x}_1} \bullet \alpha_1(p_1, s) ds + \dots + \int_{\bar{x}_{n-1}}^v \bullet \alpha_n(p_n, s) ds \right].$$

The inside of the brackets of the right hand side of this equality is a smooth function of $u, \bar{x}_1, \dots, \bar{x}_{n-1}, v, p_1, \dots, p_n$. Taking the standard parts (that, as usual, includes the use of the dominated convergence theorem for $t \rightarrow 0^+$ in \mathbb{R}) and using (20), we obtain that

$$\begin{aligned} \circ \left(\int_u^v f \right) &= \int_{\circ u}^{\bar{x}_1} \alpha_1(\circ p_1, s) \, ds + \dots + \int_{\bar{x}_{n-1}}^{\circ v} \alpha_n(\circ p_n, s) \, ds \\ &= \int_{\circ u}^{\circ v} \circ f \\ &= \int_{\circ u}^{\bar{x}_1} \alpha_1(\circ p_1, s) \, ds + \dots + \int_{\bar{x}_{n-1}}^{\circ v} \alpha_n(\circ p_n, s) \, ds \\ &= \int_{\circ u}^{\circ v} \circ f. \end{aligned}$$

Now, we can compute the infinitesimal part by

$$\begin{aligned} \delta \left(\int_u^v f \right) &= \int_u^v f - \int_{\circ u}^{\circ v} \circ f \\ &= \int_u^{\circ u} (\circ f + \delta f) + \int_{\circ u}^{\circ v} (\circ f + \delta f) + \int_{\circ v}^v (\circ f + \delta f) - \int_{\circ u}^{\circ v} \circ f \\ &= \int_{\circ u + \delta u}^{\circ u} \circ f + \int_{\circ v}^{\circ v + \delta v} \circ f + \int_u^v \delta f \\ &= - \sum_{j=1}^{[\omega(u)]} \frac{\sigma f^{(j-1)}(\circ u)}{j!} (\delta u)^j + \sum_{i=1}^{[\omega(v)]} \frac{\sigma f^{(i-1)}(\circ v)}{i!} (\delta v)^i + \int_u^v \delta f, \end{aligned}$$

where, in the last step, we have used the infinitesimal Taylor's formula (2) for (the extension of) ordinary smooth functions. \square

The standard and infinitesimal parts of general Fermat spaces will be discussed in [15, Sec. 4.2].

Example: Divergence and curl

Now we use suitable infinitesimals in Fermat reals to revisit the classical concepts of divergence and curl.

Traditionally, $\text{div} \vec{A}(x)$ is understood as the density of the flux of a vector field $\vec{A} \in \mathcal{C}^\infty(U, \mathbb{R}^3)$ through an “infinitesimal parallelepiped” centered at the point x in an open set $U \subseteq \mathbb{R}^3$. To formalize this concept, we take three vectors $\vec{h}_1, \vec{h}_2, \vec{h}_3 \in {}^\bullet\mathbb{R}^3$ and express them with respect to a fixed base $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^3$ as

$$\vec{h}_i = k_i^1 \cdot \vec{e}_1 + k_i^2 \cdot \vec{e}_2 + k_i^3 \cdot \vec{e}_3, \quad \text{where } k_i^j \in {}^\bullet\mathbb{R}.$$

We say that $P := (x, \vec{h}_1, \vec{h}_2, \vec{h}_3)$ is a *(third order) infinitesimal parallelepiped* if

$$x \in \mathbb{R}^3 \text{ and each } k_i^j \in D_3.$$

Here the requirement that each $k_i^j \in D_3$ is to make sure that the multiplication of any four (and hence more) such k_i^j 's is zero. The flux of the vector field \vec{A} through such a parallelepiped (toward the outer) is, by definition, the sum of the fluxes through every “face”:

$$\begin{aligned} \int_P \vec{A} \cdot \vec{n} \, dS &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet \vec{A}(x - \frac{1}{2}\vec{h}_3 + t\vec{h}_1 + s\vec{h}_2) \cdot (\vec{h}_2 \times \vec{h}_1) \, ds + \\ &+ \int_{-\frac{1}{2}}^{\frac{1}{2}} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet \vec{A}(x + \frac{1}{2}\vec{h}_3 + t\vec{h}_1 + s\vec{h}_2) \cdot (\vec{h}_1 \times \vec{h}_2) \, ds + \dots, \end{aligned}$$

where the fluxes through the two opposite faces spanned by \vec{h}_1 and \vec{h}_2 are explicit in the above formula, \times is the cross product, and the dots \dots indicate similar terms for the other faces of the parallelepiped. Let us note that e.g. the function $s \mapsto \bullet \vec{A}(x - \frac{1}{2}\vec{h}_3 + t\vec{h}_1 + s\vec{h}_2)$ is a quasi-standard smooth function. In this case, the fixed parameter is $p = (x, t, \vec{h}_1, \vec{h}_2, \vec{h}_3) \in \bullet \mathbb{R}^{13}$. It is easy to prove that if $\vec{A} \in \mathcal{C}^\infty(U, \mathbb{R}^3)$ and the oriented volume $\text{Vol}(\vec{h}_1, \vec{h}_2, \vec{h}_3) \in \bullet \mathbb{R}^{13}$ of the infinitesimal parallelepiped $P = (x, \vec{h}_1, \vec{h}_2, \vec{h}_3)$, is not zero, then the following ratio between infinitesimals exists and is independent of $\vec{h}_1, \vec{h}_2, \vec{h}_3$ (see [9] for the notion of ratio between infinitesimals):

$$\text{div} \vec{A}(x) := \frac{1}{\text{Vol}(\vec{h}_1, \vec{h}_2, \vec{h}_3)} \cdot \int_P \vec{A} \cdot \vec{n} \, dS \in \mathbb{R}.$$

To define the curl of a vector field $\vec{A} \in \mathcal{C}^\infty(U, \mathbb{R}^3)$, we say that $C := (x, \vec{h}_1, \vec{h}_2)$ is a *(second order) infinitesimal cycle* if

$$x \in U \text{ and each } k_i^j \in D_2.$$

Here the requirement that each $k_i^j \in D_2$ is to make sure that the multiplication of any three (and hence more) such k_i^j 's is zero. The circulation of the vector field \vec{A} on this cycle C is defined as the sum of the “line integrals” over each “side”:

$$\int_C \vec{A} \cdot \vec{t} \, dl := \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet \vec{A}(x - \frac{1}{2}\vec{h}_2 + t\vec{h}_1) \cdot \vec{h}_1 \, dt - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet \vec{A}(x + \frac{1}{2}\vec{h}_2 + t\vec{h}_1) \cdot \vec{h}_1 \, dt + \dots,$$

where the line integrals of the two opposite sides spanned by \vec{h}_1 are explicit in the above formula, and the dots \dots indicate similar terms for the other sides of the cycle C (noting the consistency of the orientation on each side). Once again, using exactly the calculations frequently done in elementary

courses of physics, one can prove that there exists one and only one vector, $\text{curl } \vec{A}(x) \in \mathbb{R}^3$, such that

$$\int_C \vec{A} \cdot \vec{t} \, dl = \text{curl } \vec{A}(x) \cdot (\vec{h}_1 \times \vec{h}_2)$$

for every infinitesimal cycle $C = (x, \vec{h}_1, \vec{h}_2)$, representing thus the (vector) density of the circulation of \vec{A} .

8. Inequalities for integrals

In this section, we derive similar inequalities for integrals of smooth functions to the classical ones for integrals of ordinary smooth functions. Thanks to the total order on $\bullet\mathbb{R}$, we are able to prove the monotonicity of integrals, a replacement of the inequality for absolute value of smooth functions, and the Cauchy-Schwarz inequality, with respect to arbitrary integration interval. The strategy is, we first prove inequality for real integration interval, and then use integration by substitution for arbitrary integration interval.

Recall (see e.g. [10]) that if $x, y \in \bullet\mathbb{R}$, then $x \leq y$ if and only if for any $x_t, y_t \in \mathbb{R}_o[t]$ with $x = [x_t]$ and $y = [y_t]$, there exists $z_t \in \mathbb{R}_o[t]$ such that $[z_t] = 0$ in $\bullet\mathbb{R}$ and $x_t \leq y_t + z_t$ for $t \in \mathbb{R}_{\geq 0}$ sufficiently small. In proving these integral inequalities, we will use [10, Thm. 28]: if $(x_t)_t$ is any little-oh polynomial representing $x \in \bullet\mathbb{R}$, then $x \geq 0$ is equivalent to $x_t \geq 0$ for t sufficiently small, or $x = 0$ in $\bullet\mathbb{R}$ (i.e., $x_t = o(t)$ as $t \rightarrow 0^+$ in \mathbb{R}).

The first two results concerns the monotonicity of integrals:

Lemma 28. *Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$, and let $f, g : J \rightarrow \bullet\mathbb{R}$ be smooth functions. Let $a, b \in J \cap \mathbb{R}$, with $a < b$. Then the assumption*

$$g(x) \leq f(x) \quad \forall x \in [a, b]_{\mathbb{R}} \quad (28)$$

implies $\int_a^b g \leq \int_a^b f$.

Proof. It suffices to prove the conclusion for $g = 0$. By Lem. 16, we can write

$$\int_a^b f = \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} \bullet\alpha_i(p_i, s) \, ds,$$

where $\alpha_i \in \mathcal{C}^\infty(U_i \times V_i, \mathbb{R})$, $U_i \subseteq \mathbb{R}^{p_i}$, $p_i \in \bullet U_i$, $V_i = (x_i - \delta_i, x_i + \delta_i)_{\mathbb{R}}$, $\delta_i \in \mathbb{R}_{>0}$, $\bar{x}_i \in (x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}} = V_i \cap V_{i+1} \subset (x_i, x_{i+1})_{\mathbb{R}}$, and

$$f(x) = \bullet\alpha_i(p_i, x) \quad \forall x \in \bullet V_i \cap J, \quad (29)$$

$$a = \bar{x}_0 = x_1 < x_2 < \dots < x_n = \bar{x}_n = b.$$

For each $i = 1, \dots, n$ and each $s \in [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}}$, we have $a = \bar{x}_0 \leq \bar{x}_{i-1} \leq s \leq \bar{x}_i \leq \bar{x}_n = b$, so $s \in [a, b]_{\mathbb{R}}$. Moreover, $\bar{x}_{i-1} \in V_i$ and $\bar{x}_i \in V_i$ yield that

$x_i - \delta_i < \bar{x}_{i-1} \leq s \leq \bar{x}_i < x_i + \delta_i$, so $s \in V_i \cap J$. Thereby, assumption (28) and equality (29) imply that $f(s) = \bullet \alpha_i(p_i, s) \geq 0$. The mentioned result [10, Thm. 28] yields that

$$\begin{aligned} \forall^0 t \geq 0 : \alpha_i((p_i)_t, s) &\geq 0 \\ \text{or} \\ \bullet \alpha_i(p_i, s) &= 0 \text{ in } \bullet \mathbb{R}. \end{aligned} \quad (30)$$

Set $w_i(t, s) := 0$ if $\alpha_i((p_i)_t, s) \geq 0$ and $w_i(t, s) := \alpha_i((p_i)_t, s)$ otherwise. Then,

$$\forall t \in \mathbb{R}_{\geq 0} \forall s \in [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}} : \alpha_i((p_i)_t, s) \geq w_i(t, s), \quad (31)$$

and from (30) we get that $w_i(t, s) = o(t)$ as $t \rightarrow 0^+$ in \mathbb{R} . Note also that $w_i(t, -) : [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}} \rightarrow \mathbb{R}$ is integrable by definition, so that (31) implies

$$\int_{\bar{x}_{i-1}}^{\bar{x}_i} \alpha_i((p_i)_t, s) ds \geq \int_{\bar{x}_{i-1}}^{\bar{x}_i} w_i(t, s) ds \quad \forall t \in \mathbb{R}_{\geq 0}. \quad (32)$$

But Lebesgue dominated convergence theorem and the fact that $w_i(t, s) = o(t)$ yield

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \cdot \int_{\bar{x}_{i-1}}^{\bar{x}_i} w_i(t, s) ds = \int_{\bar{x}_{i-1}}^{\bar{x}_i} \lim_{t \rightarrow 0^+} \frac{w_i(t, s)}{t} ds = 0, \quad (33)$$

i.e., the right hand side of the inequality (32) viewed as a function of t is $o(t)$ as $t \rightarrow 0^+$ in \mathbb{R} , and hence it is in $\mathbb{R}_o[t]$ and it represents 0 in $\bullet \mathbb{R}$. In other words, we have proved that

$$\int_{\bar{x}_{i-1}}^{\bar{x}_i} \bullet \alpha_i(p_i, s) ds \geq 0.$$

We thus have

$$\int_a^b f = \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} \bullet \alpha_i(p_i, s) ds \geq 0,$$

which proves our conclusion. We finally note that in (33), all the endpoints of the integration intervals, which include $a = \bar{x}_0$ and $b = \bar{x}_n$, do not depend on t . \square

We can now extend this result to the case where the endpoints a, b of the integration interval are arbitrary Fermat reals.

Theorem 29. *Let J be a non-infinitesimal interval of $\bullet \mathbb{R}$, and let $f, g : J \rightarrow \bullet \mathbb{R}$ be smooth functions. Let $a, b \in J$, with $a < b$. Then the assumption*

$$g(x) \leq f(x) \quad \forall x \in [a, b] \quad (34)$$

implies $\int_a^b g \leq \int_a^b f$.

Note that in this theorem (also the next two theorems), we include the case when the integration interval is infinitesimal, i.e., ${}^\circ a = {}^\circ b$.

Proof. As above, without loss of generality, we may assume $g = 0$. Using integration by substitution (Thm. 19), we can write

$$\int_a^b f = (b - a) \cdot \int_0^1 f[a + (b - a)\sigma] d\sigma \quad (35)$$

For each $\sigma \in [0, 1]_{\mathbb{R}}$, we have $a \leq a + (b - a)\sigma \leq b$. Note that $a + (b - a)\sigma$ is in general not standard if a is not standard, which justifies the stronger assumption (34) compared to the previous (28). Assumption (34) yields that $f[a + (b - a)\sigma] \geq 0$. Lem. 28 implies that the right hand side of the integral in (35) is non-negative, from which the conclusion follows. \square

Although we haven't justified the validity of integrals of absolute values of smooth functions, the following result is a possible substitute of the property $\left| \int_a^b f \right| \leq \int_a^b |f|$ for classical integrals.

Theorem 30. *Let J be a non-infinitesimal interval of ${}^\bullet\mathbb{R}$, and let $f, g : J \rightarrow {}^\bullet\mathbb{R}$ be smooth functions. Let $a, b \in J$, with $a < b$. Then the assumption*

$$|f(x)| \leq g(x) \quad \forall x \in [a, b] \quad (36)$$

implies $\left| \int_a^b f \right| \leq \int_a^b g$.

Proof. The proof is conceptually identical to those of Lem. 28 and Thm. 29. We first prove the conclusion for $a, b \in \mathbb{R}$. The interval $[a, b]_{\mathbb{R}}$ can be covered by a finite number of intervals $X_i := (x_i - \delta_i, x_i + \delta_i)_{\mathbb{R}}$, where we can write both $f(x) = {}^\bullet\alpha_i(p_i, x)$ and $g(x) = {}^\bullet\beta_i(q_i, x)$ for every $x \in {}^\bullet X_i \cap [a, b]$. Furthermore, we have

$$\begin{aligned} \int_a^b f &= \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} {}^\bullet\alpha_i(p_i, s) ds, \\ \int_a^b g &= \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} {}^\bullet\beta_i(q_i, s) ds, \end{aligned}$$

where $\bar{x}_i \in (x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}$ and $a = \bar{x}_0 = x_1 < x_2 < \dots < x_n = \bar{x}_n = b$. For each $i = 1, \dots, n$ and each $s \in [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}}$, assumption (36) and [10, Thm. 28] yield that

$$\forall {}^0t \in \mathbb{R}_{\geq 0} : |\alpha_i((p_i)_t, s)| \leq \beta_i((q_i)_t, s)$$

or

$$|{}^\bullet\alpha_i(p_i, s)| = {}^\bullet\beta_i(q_i, s) \text{ in } {}^\bullet\mathbb{R}.$$

Set

$$w_i(t, s) := \begin{cases} 0, & \text{if } |\alpha_i((p_i)_t, s)| \leq \beta_i((q_i)_t, s) \\ |\alpha_i((p_i)_t, s)| - \beta_i((q_i)_t, s), & \text{otherwise.} \end{cases}$$

Therefore,

$$\forall t \in \mathbb{R}_{\geq 0} \forall s \in [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}} : |\alpha_i((p_i)_t, s)| \leq \beta_i((q_i)_t, s) + w_i(t, s). \quad (37)$$

As in the proof of Lem. 28, one can show that for any fixed $s \in [\bar{x}_{i-1}, \bar{x}_i]_{\mathbb{R}}$, $w_i(t, s) = o(t)$, and for any fixed $t \in \mathbb{R}_{\geq 0}$, the function $w_i(t, -) : [\bar{x}_{i-1}, \bar{x}_i] \rightarrow \mathbb{R}$ is integrable. Again, one gets $\int_{\bar{x}_{i-1}}^{\bar{x}_i} w_i(t, s) ds = o(t)$, and as a consequence, $\int_{\bar{x}_{i-1}}^{\bar{x}_i} w_i(t, s) ds \in \mathbb{R}_o[t]$ and it represents 0 in $\bullet\mathbb{R}$. From inequality (37) we get the following inequality of little-oh polynomials:

$$\begin{aligned} \left| \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} \alpha_i((p_i)_t, s) ds \right| &\leq \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} |\alpha_i((p_i)_t, s)| ds \\ &\leq \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} \beta_i((q_i)_t, s) ds + \sum_{i=1}^n \int_{\bar{x}_{i-1}}^{\bar{x}_i} w_i(t, s) ds. \end{aligned}$$

Since the last summand in the above formula represents zero in $\bullet\mathbb{R}$, this proves our conclusion for $a, b \in \mathbb{R}$.

Now, we can proceed as in the proof of Thm. 29:

$$\begin{aligned} \left| \int_a^b f \right| &= (b-a) \cdot \left| \int_0^1 f[a + (b-a)\sigma] d\sigma \right| \\ &\leq (b-a) \cdot \int_0^1 g[a + (b-a)\sigma] d\sigma \\ &= \int_a^b g. \end{aligned}$$

□

We have just showed that Thm. 30 follows from Lem. 16 and the analogous property of the Riemann integral. In the same way, we can prove the following:

Theorem 31 (Cauchy-Schwarz inequality). *Let J be a non-infinitesimal interval of $\bullet\mathbb{R}$, and let $f, g : J \rightarrow \bullet\mathbb{R}$ be smooth functions. Let $a, b \in J$, with $a < b$. Then $(\int_a^b fg)^2 \leq (\int_a^b f^2) \cdot (\int_a^b g^2)$.*

9. Multiple integrals and Fubini's theorem

In previous sections, we have developed integral calculus with an interval as integration domain. In this section we will define the integral (and multiple integral) of a smooth function with a more general integration domain. The approach is similar to that used for the Peano-Jordan measure.

We denote by \mathcal{I} the set of all the intervals of $\bullet\mathbb{R}$ (see Sec. 2). Thanks to the total order relation \leq on $\bullet\mathbb{R}$, every interval uniquely determines its endpoints. (We recall that this property is not true e.g. in SDG, where \leq is a pre-order.) Since the domain of every smooth function defined on an interval of $\bullet\mathbb{R}$ can always be extended to a Fermat open subset (Lem. 8), we can therefore define the integral over an interval as follows:

Definition 32. Let J be a non-infinitesimal interval, let $f : J \rightarrow \bullet\mathbb{R}$ be a smooth function, and let $I \in \mathcal{I}$ be an interval contained in J . We define

$$\int_I f := \int_{\inf(I)}^{\sup(I)} f.$$

The following lemma permits to order the endpoints of pairwise disjoint intervals.

Lemma 33. *If $I, J \in \mathcal{I}$ are disjoint intervals, then either $\sup(I) \leq \inf(J)$ or $\sup(J) \leq \inf(I)$.*

Proof. Set $a := \inf(I)$, $b := \sup(I)$, $\bar{a} := \inf(J)$ and $\bar{b} := \sup(J)$. By contradiction, we assume $\bar{a} < b$ and $a < \bar{b}$. We distinguish four cases: $a \leq \bar{a}$ and $\bar{b} \leq b$, $a \leq \bar{a}$ and $b \leq \bar{b}$, $\bar{a} \leq a$ and $\bar{b} \leq b$, $\bar{a} \leq a$ and $b \leq \bar{b}$. In each case, we can find a point in $I \cap J$. \square

This lemma permits to integrate smooth functions over a finite pairwise disjoint union of intervals:

Theorem 34. *Let $\{I_1, \dots, I_n\}$ and $\{J_1, \dots, J_m\}$ be two finite pairwise disjoint families of \mathcal{I} such that*

$$\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m J_j =: U.$$

Let J be a non-infinitesimal interval and let $f : J \rightarrow \bullet\mathbb{R}$ be a smooth function. Then

$$\sum_{i=1}^n \int_{I_i} f = \sum_{j=1}^m \int_{J_j} f =: \int_U f. \quad (38)$$

Proof. Set $a_i := \inf(I_i)$, $b_i := \sup(I_i)$. Thanks to Lem. 33, we can always rearrange the indices i, j so that

$$a_1 \leq b_1 \leq \dots \leq a_n \leq b_n.$$

Since the intervals are pairwise disjoint, an equality of the type $b_i = a_{i+1}$ may happen only if $b_i \notin I_i$ or $a_{i+1} \notin I_{i+1}$. If $I_i \cup I_{i+1}$ is connected, we could replace I_i and I_{i+1} in the set $\{I_1, \dots, I_n\}$ by $I_i \cup I_{i+1}$ to get another finite pairwise disjoint family $\{I'_1, \dots, I'_{n-1}\}$ of \mathcal{I} . In this case, because $\int_{I_i} f + \int_{I_{i+1}} f = \int_{a_i}^{b_{i+1}} f = \int_{I_i \cup I_{i+1}} f$, we know that

$$\sum_{i=1}^n \int_{I_i} f = \sum_{i=1}^{n-1} \int_{I'_i} f.$$

In other words, we can combine two “joinable” intervals to get a new set of finite pairwise disjoint family, and keep the integral unchanged. After finitely many such procedures, we reach a set of finite pairwise disjoint family consisting of all the connected components of U , and by the invariance of integrals, we get

$$\sum_{i=1}^n \int_{I_i} f = \sum \left\{ \int_I f \mid I \text{ is a connected component of } U \right\}.$$

We arrive at the same result by considering $\sum_{j=1}^m \int_{J_j} f$. \square

In the following definition, we identify the general domains for integrals of higher dimensions.

Definition 35. A *box* in $\bullet\mathbb{R}^d$ is a set of the form $B = \prod_{i=1}^d I_i$, where $I_i \in \mathcal{I}$ are intervals. We denote by \mathcal{B}^d the set of all the boxes in $\bullet\mathbb{R}^d$. An *elementary set* $U \in \mathcal{E}$ is a set of the form $\bigcup_{i=1}^n B_i$, where $B_i \in \mathcal{B}^d$ are boxes.

Theorem. \mathcal{E} is a Boolean algebra with respect to union, intersection and set difference.

Proof. It is enough to show that \mathcal{E} is closed under these operations. The closure of \mathcal{E} with respect to union is clear. If $U = \bigcup_{i=1}^n B_i$ and $V = \bigcup_{k=1}^m C_k$, where $B_i = \prod_{j=1}^d I_{ij}$ and $C_k = \prod_{j=1}^d J_{kj}$ are boxes, then

$$U \cap V = \bigcup \left\{ \prod_{j=1}^d I_{ij} \cap J_{kj} \mid i = 1, \dots, n, j = 1, \dots, m \right\}.$$

Since $I_{ij} \cap J_{kj} \in \mathcal{I}$ (including the empty interval), $U \cap V \in \mathcal{E}$. To prove the closure of \mathcal{E} with respect to set difference, we first introduce the notion

of complement $A^c := \bullet\mathbb{R}^d \setminus A$ for any subset $A \subseteq \bullet\mathbb{R}^d$. Using the same notations, since

$$C_k^c = \left(\prod_{j=1}^d J_{kj} \right)^c = \cup \left\{ \prod_{i=1}^{l-1} \bullet\mathbb{R} \times (J_{kl}^c)^s \times \prod_{i=l+1}^d \bullet\mathbb{R} \mid l = 1, \dots, d, s = \pm \right\} \in \mathcal{E},$$

where $(J_{kl}^c)^+ = \{x \in \bullet\mathbb{R} \mid x > y \ \forall y \in J_{kl}\}$ and $(J_{kl}^c)^- = \{x \in \bullet\mathbb{R} \mid x < y \ \forall y \in J_{kl}\}$. we know that $V^c = (\cup_{k=1}^m C_k)^c = \cap_{k=1}^m C_k^c \in \mathcal{E}$, and hence $U \setminus V = U \cap V^c \in \mathcal{E}$. \square

In our framework, the following result corresponds to Fubini's theorem.

Theorem 36. *Let $B = \prod_{i=1}^d I_i \in \mathcal{B}^d$ be a box in $\bullet\mathbb{R}^d$. Let $U \subseteq \bullet\mathbb{R}^d$ be a Fermat open set such that $B \subseteq U$, and let $f : U \rightarrow \bullet\mathbb{R}$ be a smooth function. Let $(\sigma_1, \dots, \sigma_d)$ be a permutation of $\{1, \dots, d\}$. Then*

$$\begin{aligned} \int_{I_1} \dots \left(\int_{I_d} f(x_1, \dots, x_d) dx_d \right) \dots dx_1 \\ = \int_{I_{\sigma_1}} \dots \left(\int_{I_{\sigma_d}} f(x_1, \dots, x_d) dx_{\sigma_d} \right) \dots dx_{\sigma_1}. \end{aligned}$$

We denote this integral by $\int_B f$.

Proof. We prove only for the case $d = 2$, since the general case is similar. Let a, b, \bar{a}, \bar{b} be the endpoints of I_1 and I_2 , respectively. Set $F(t) := \int_{\bar{a}}^{\bar{b}} f(t, y) dy$ and $G(t) := \int_{\bar{a}}^{\bar{b}} \left(\int_a^t f(x, y) dx \right) dy$ for each $t \in I_1$. Both F and G are smooth functions such that $G(a) = 0$ and

$$G'(t) = \int_{\bar{a}}^{\bar{b}} \frac{\partial}{\partial t} \left(\int_a^t f(x, y) dx \right) dy = \int_{\bar{a}}^{\bar{b}} f(t, y) dy = F(t) \quad \forall t \in I_1.$$

Therefore, $G = \int_a^{(-)} F$ and $G(b) = \int_{\bar{a}}^{\bar{b}} \left(\int_a^b f(x, y) dx \right) dy = \int_a^b F(t) dt = \int_a^b \left(\int_{\bar{a}}^{\bar{b}} f(x, y) dy \right) dx$, which is our conclusion. \square

Similarly to Thm. 34, we can finally prove the following:

Theorem 37. *Let $\{B_1, \dots, B_n\}$ and $\{C_1, \dots, C_m\}$ be two finite pairwise disjoint families of \mathcal{B}^d such that*

$$\bigcup_{i=1}^n B_i = \bigcup_{j=1}^m C_j =: U.$$

Let V be a Fermat open subset of $\bullet\mathbb{R}^d$ containing U , and let $f : V \rightarrow \bullet\mathbb{R}$ be a smooth function. Then

$$\sum_{i=1}^n \int_{B_i} f = \sum_{j=1}^m \int_{C_j} f =: \int_U f. \quad (39)$$

10. Conclusions

The main aim of the present work is to lay the foundation of the integral calculus for smooth functions defined on the ring of Fermat reals. In the Introduction 1, we have already summarized the major achievements of this theory compared to other non-Archimedean theories. We can outline them by saying: better order properties and easier to deal with infinite dimensional integral operators.

The present work confirms that the ring of Fermat reals does not represent a new foundation of the calculus. On the other hand, the definition of integral as primitive (Def. 14) and the corresponding simple “global proofs” (see e.g. Thm. 17) can suggest ideas for a new approach to a subset of the classical integral calculus. On these bases, we can also think of the possibility to approach the integral calculus of smooth functions from an axiomatic point of view.

Finally, our work shows that all the classical instruments to deal with integrals of smooth functions with infinitesimal parameters are now available. This is important for the applications in Physics.

A suitable notion of convergence for sequences of smooth functions, the exchange between limit and integral, and a more general class of integration domains remain open. We plan to explore them in a future work.

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